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AN INTEGER PROGRAMMING APPROACH TO CAPACITY EXPANSION AND PRODUCTION PLANNING

BY

RICHARD H. BARCHI

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Baltimore, Maryland 21218

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Abstract

The combined capacity expansion, production-inventory problem is modeled as a linear, integer program and then extended to also allow capacity decrease. The models assume constant returns to scale in the production function of a firm which must meet deterministic demands for a single product over N periods, with no backordering, at minimum cost.

A linear transformation is used to obtain equivalent forms of the models which are then decomposed into fixed cost and variable cost parts. Enumeration is done on the fixed cost variables and transportation sub-problems are solved with the remaining variables. Special demand and cost structures are investigated, one of which leads to formulation of the entire problem as a network.

Extensions are made to include piecewise-linear, convex production costs and lumpiness in capacity acquisition. A reinterpretation of the models to handle the employment smoothing problem is also presented. Finally, computational results of a Fortran IV coding of the basic algorithms are presented.

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Chapter I

I.1 Introduction

The quantitative approach to the solution of problems in the realm of managerial economics can be traced back to the efforts of the early industrial engineers in the United States at the end of the nineteenth century. These men, most notably Frederick W. Taylor, pioneered the industrial engineering concepts such as job description, time-motion studies, etc., which had such a great influence on the development of the assembly line in industry during the first half of the twentieth century.

With World War II, however, an entirely new set of tools was refined and applied by the early practitioners in the infant field of Operations Research. Though most of the techniques had their origin in military applications, they gradually found their way into the industrial sphere in the peace-time economic expansion resulting from the war. In the intervening decades, the complexity of the practical problems solved has moved apace with the rapid advances in theory typical of a new field. The logical result is that an increasing amount of effort has been put into inte-

gration of past techniques using newer theory in an attempt to provide more realistic approaches to problems with broader horizons.

The present thesis makes such an attempt by offering models of the combined production-inventory and capacity expansion problem and algorithms which give all integer solutions to these models. Discussion of the importance of the combined problem and the integral approach are given later in this chapter. This is the first integer approach to the combined problem known to exist.

The basic technique used will be branch and bound with network sub-problems. A unique feature of the algorithm is the capability of starting it at any point in the enumeration. This is especially meaningful when linear programming is used to give a universal, and often non-integer, optimal solution. The enumeration can be started from a suitable point with respect to the linear programming optimum to determine the existence of any optimal, rounded-off solutions rather quickly.

The famous economic problem of "lumpiness" in capacity acquisition as well as multi-shift, and overtime, operation of the production facility will be treated. A discussion of the relationship between

the traditional capacity expansion and employment smoothing literatures and re-interpretation of the results in terms of employment smoothing will also be given.

I.2 Problem Description

A useful paradigm is a company with a single producing facility and a homogeneous product faced with a schedule of deterministic demands over a set of discrete periods to some pre-set horizon. These demands must be met on time from production or existing inventory at minimum possible cost.

The capacity of the production facility is defined as its output in units per time period when equipment is being run at its maximum efficient rate with a single shift not working overtime. The models will be extended to allow multiple shifts and overtime work, but note that these additions are not germane to the basic definition of capacity.

The company's basic problem, assuming that cumulative demand never exceeds cumulative capacity (where cumulative capacity now includes all regular and overtime production), is to find a production-inventory schedule to meet demand at minimum

cost.[1] References to this problem are ubiquitious in both the operations research literature and the economics literature, and will not be discussed in this work.

If, however, cumulative demand does exceed cumulative capacity within the planning horizon, then the company must make a capacity expansion decision. That is, it must decide at what time (or times) and by what amount (or amounts) to invest in capital equipment so that production can satisfy demand. The intuitive difficulty of this problem is the fact that a change in the capacity in one period affects the production-inventory schedule for all future periods, thus lending combinatorial aspects to the capacity expansion problem. Further difficulty arises when capacity must be added in indivisible units, each capable of producing large numbers of units of output as is usually the case in industry.

Capital equipment itself has many aspects. Deterioration might be a significant factor with the

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[1] The existence of capacity in excess of demand does not preclude a decision to expand capacity since there could exist circumstances where such a move would lead to lower costs.

type of capital installed, in which case the capacity expansion problem is further complicated by the need for equipment replacement. On the other hand, fixed capital might have a relatively long life, making periodic maintenance the only significant factor within the time-span of interest. It is also possible that the fixed capital equipment can be salvaged or scrapped if warranted by changes in the demand situation. The first case can only be approximated by the models which follow and then only by allowing maintenance costs to reflect the increased investment over time needed to maintain equipment at its original capacity. In addition to not obtaining equipment replacement decisions from the models, the approximation is faulty in that the increased maintenance costs are applied uniformly to equipment in each period, regardless of age. The latter two cases are modeled exactly, however, the second by the explicit structure of the maintenance costs and the last implicitly by the relevant mathematical programming model.

I.3 Economic Background

With respect to the economic geography, the

problem lies on the boundary of capital theory and the classical theory of production. A good discussion of the problems arising from the merger of these two disciplines is contained in the unpublished Ph. D. thesis of Carroll (Carroll 68).

Apropos of capital theory, a decision must be made as to the proper points in time, and the proper amounts, for investment in capacity; however, there is no requirement present for alternative investments. In general, if expansion of capacity in a number of different areas were being considered in each period with a limited budget, then the extensive work in capital budgeting would be relevent.[2] Further, capital budgeting problems typically abstract from the micro problem of optimizing the form of each investment project, being concerned mainly with the macro allocation of funds to projects. The micro-optimization is an important part of the work that follows. Extensions of the present problem to the capital budgeting sphere will be discussed in the final chapter.

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[2] A classical application of mathematical programming to the capital budgeting problem is given by Weingartner (Weingartner 63).

In the theory of production as discussed by Carroll, consideration of production, inventory, and capacity expansion places the problem in between the short term, where capacity is fixed and decisions are only made on levels of production, and the long-run decision of optimal overall capacity level, which abstracts from all short-run decisions. In effect, the mechanism of inventorying production functions as the mediator between the short-run minimum cost production problems and the long-run minimum cost capacity expansion problems, serving to "smooth" the extremes in the optimal solutions to the individual problems.

A more satisfying concept of the spectrum of decision time-spans is given by Maxwell (Maxwell 65). He considers the very short-run as the time-span in which all factors of production are fixed in both acquisition and use. Successive decision time-spans are encountered as factors become variable in use and acquisition, giving a succession of short runs of increasing length. The long-run is finally achieved when all factors become variable in both use and acquisition. Against this spectrum, the problem at hand ranges from a succession of relatively long short-runs when only capital stock is fixed in acquisition and all factors are variable in use, to a long-run over which the

capital stock is variable in acquisition at a succession of discrete points. Correspondence with classical theory is not exact though, since capital is durable and not perfectly flexible as required in the classical long-run. The necessity for providing inter-period relationships through the mechanism of inventorying in the case of inflexible capital was pointed out by Carroll:

> ...So long as the theory of production abstracted from the particular problems of fixed in-puts, it was essentially a mono-period theory in which inter-period relationships could largely be neglected. Even the long-run analyses tended to treat the question of determining the optimal capital stock as a once-and-for-all decision. Since capital was generally assumed to be completely flexible in the long-run, questions of the impact of one period's capital accumulation decision upon that of another were not raised. But once capital came to be viewed as a durable good, which was not perfectly flexible, inter-period (i.e., between long-run periods) relationships were a matter of interest. (1968, p. 5)

In the following work, a salvage value of capital goods is introduced and constrained to be less than the acquisition cost, thus ensuring at least a mild flexibility in capital.

Maxwell also presents a variation on the conventional theory of the firm which is relevant here. The contrast is in the derivation of the production

function which, in the classical theory, is obtained from the combination of varying inputs of consumptive factors of production with the fixed factor to obtain a graph of maximum outputs over the range of the consumptive factors. The underlying assumption is, of course, that the fixed factor is indivisible in use and must be used to the available limit. If instead, the fixed factor is viewed as indivisible in acquisition, but divisible in use, a more useful approach to the firm's production function can be derived.

Assume constant returns to scale in the production function. Then, for each level of input of consumptive factors, the fixed factor is used up to its zero level of marginal productivity giving a linear expansion path for the firm up to the capacity of the fixed factor. Thus, under constant returns, the scale of use of the fixed and consumptive factors of production is varied linearly up to the capacity of the fixed factor, while the proportions of both remain constant. Beyond the capacity of the fixed factor the proportion of the consumptive factor is increased, giving returns to proportion in production as opposed to the returns to scale obtained in the former range. The importance of this concept to the theory of the firm is pointed out by Maxwell:

> Many empirical cost studies indicate that the marginal costs tend to be constant very nearly up to capacity, suggesting that this range in which less than all of the stock of fixed-factor services is being utilized may indeed be a significant portion of the total range of outputs that the firm is capable of producing in the short-run. (1965, p. 106)

For the problem of interest, the range of the production function reflecting returns to proportion is of no interest since one of the decision variables for each period is the optimal level of fixed capital for that period. Thus, concern can be focused solely on the constant returns to scale portion of the production function assumed present up to the capacity of the fixed factor.

In the problem which follows, the bundle of consumptive factors of production ∂_i which is needed in period i for one unit of output is assumed known. Further, it is assumed that the fixed factor of production is used at its zero marginal productivity level for this bundle ∂_i or any multiple of it. The capacity W_i of the fixed factor in any period can then also be expressed in terms of the maximum number of bundles ∂_i which can be used efficiently with the fixed factor in the period (in this case the period includes only regular time work by one shift). The level of production

in period i which is x_i must thus satisfy

$$x_i \leq W_i \qquad i = 1,\ldots,N$$

These assumptions, when taken in the context of fixed capital investment, suppress the problem of labor force. That is, although the labor force is a non-consumptive factor of production, the necessary assumption here is that labor force is always adjusted to the exact size required to run the capital equipment at the predetermined optimal level x_i in period i. An interpretation of the problem of capacity in terms of labor force rather than capital equipment will be given subsequently.

The objective is to minimize the present value of the costs incurred over N periods to meet a predetermined demand schedule. It should be noticed that the problem is abstracted from pricing considerations and from determination of optimum output schedules; in fact, concern is with the optimum method of meeting any demand schedule. To achieve the present value criterion, it is only necessary that the individual costs in the objective function of the model be discounted by the appropriate cost of capital ρ_i in period i. Thus discounting does not have to be modeled

but can be introduced explicitly in the relevant model coefficients. The subject of the proper choice of the discount factor ρ is not germane to this thesis and will not be discussed.

The major stumbling block to the merger of the capital and production theories has historically been the problem of fixed capital as a non-consumptive factor of production. This question is also lucidly discussed by Carroll. To summarize, some method must be found to measure the value of a factor of production, i.e. fixed capital, which is not consumed in the productive process, but which must be present for production to take place. In this thesis, the value measure will be the reduction in cost obtained in the production of a given demand schedule, a measure which is implicit in mathematical programming optimization.

Finally in the economic context, mention should be made of the historic bugbear of marginal analysis, indivisibility in acquisition of capital stock. This discreteness in acquisition possibilities destroys the differentiability of most useful functions at best, and their continuity itself at worst. As will be described later, a simple adjustment in the integer programming formulation will handle this problem

fairly easily and, in some cases, even improve the algorithms.

I.4 Problem Formulation

The interval of time under consideration is divided into n periods of equal lengths with the first period designated as 1 and the nth, or horizon, period designated N. It is assumed that demand in period i, d_i, is known deterministically for $i = 1,\dots,N$. Then using the following variables:

x_i- production in period i, $i = 1,\dots,N$

v_i- inventory from period i to period i + 1

w_i- capacity added in period i, $i = 1,\dots,N$

the following coefficients:

p_i- cost of a unit bundle ∂_i of consumptive factors of production in period i, $i = 1,\dots,N$

h_i- unit inventory cost from period i to i + 1

c_i- construction cost of a unit of capacity in period i

m_i- maintenance cost of a unit of capacity in period i

where coefficients are assumed integer and

$p_i \geq 0,\ h_i \geq 0,\ c_i \geq 0,\ m_i \geq 0 \quad i = 1,\dots,N$

and the following explicit integer variables:

w_o- initial capacity $(w_o \geq 0)$

v_o- initial inventory $(v_o \geq 0)$

if all functions are assumed linear, the capacity expansion-production-inventory problem (CPI) can be written as the following integer programming problem:

$$\text{P1:} \quad \min Z_1 = \sum_{i=1}^{N} [p_i x_i + h_i v_i + c_i w_i + m_i \cdot (\sum_{j=1}^{i} w_j + w_o)]$$

s.t.

$$\text{(P1.1)} \qquad x_i - \sum_{j=1}^{i} w_j \leq w_o \qquad i = 1,\ldots,N$$

$$x_1 - v_1 = d_1 - v_o$$

$$\text{(P1.2)} \qquad x_i + v_{i-1} - v_i = d_i \qquad i = 2,\ldots,N-1$$

$$x_N + v_{N-1} = d_N + v_N$$

$$\bar{x}, \bar{v}, \bar{w} \geq 0 \quad \text{and INTEGER}$$

where the right-hand side of the constraints is assumed integer. Constraints (P1.1) restrict production to be less than or equal to capacity in any period, and constraints (P1.2) require demand to be met exactly out of the production and inventory in each period. All costs are non-stationary and are explicitly discounted to the first period by the appropriate discount rate ρ_i, which is external to the model as discussed in section

I.3.

Initial inventory, v_0, and initial capacity, w_0, are set explicitly to the values relevant to the problem being solved. Initial inventory and initial capacity are rather straightforward in the problem context and no difficulty in obtaining them should occur in the real world. Final inventory, however, is more of a subjective variable. If v_N is allowed to be an implicit variable, then $v_N = 0$, since if $v_N > 0$ then the level of production in periods N, N-1, ... could be reduced until $v_N = 0$, giving a lower cost while still minimizing cost (recall that cost coefficients are non-negative). On the other hand, v_N can be useful as an explicit variable as shown by an example. Assume, for $N = 3$, that the demand schedule is $\bar{d}^T = (1,7,20)$.[3] In this case, v_N could subjectively be set to some value greater than zero as a surrogate for an expectation of continued increase in demand beyond the horizon.

The objective function in P1 can be re-

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[3] In the remainder of this thesis, an untransposed vector is assumed to be a column vector.

written as

$$\min Z_1 = \sum_{i=1}^{N}[p_i x_i + h_i v_i + (c_i + \sum_{j=i}^{N} m_j) w_i] + w_o \cdot \sum_{i=1}^{N} m_i$$

It can be clearly seen from this aggregated function, that the total cost of an incremental unit of capacity w_i is equal to its acquisition cost in period i plus the discounted sum of the maintenance costs for one unit of capacity up to the horizon. Maintenance costs themselves are period dependent (i.e., independent of the age of the installed capacity).

The CPI model can be extended easily to a capacity increase-decrease, production-inventory model (CIDPI) as shown below. Let:

r_i- capacity scrapped in period i

f_i- salvage value per unit of capacity scrapped in period i

all other variables and coefficients are as before and assume

$$f_i < c_i \qquad i = 1,\ldots,N$$

$$f_j + \sum_{k=j}^{N} m_k < c_i + \sum_{k=i}^{N} m_k \qquad \forall\ j > i,\ i = 1,\ldots,N-1$$

Then the CIDPI can be written as the following integer programming problem:

$$\text{P2:} \quad \min Z_2 = \sum_{i=1}^{N}[p_i x_i + h_i v_i + c_i w_i - f_i r_i + m_i \cdot (\sum_{j=1}^{i}(w_j - r_j) + w_o)]$$

s.t.

$$\text{(P2.1)} \quad x_i - \sum_{j=1}^{i}(w_j - r_j) \leq w_o \qquad i = 1,\ldots,N$$

$$\begin{aligned} &x_1 - v_1 = d_1 - v_o \\ \text{(P2.2)} \quad &x_i + v_{i-1} - v_i = d_i \qquad i = 2,\ldots,N-1 \\ &x_N + v_{N-1} = d_N + v_N \end{aligned}$$

$$\text{(P2.3)} \quad w_o + \sum_{j=1}^{i}(w_j - r_j) \geq 0 \qquad i = 1,\ldots,N$$

$$\bar{x},\ \bar{v},\ \bar{w},\ \bar{r} \geq 0 \quad \text{and INTEGER}$$

where the right-hand sides of the constraints are integer as before, and constraints (P2.1) and (P2.2) have the same interpretation as previously. Constraints (P2.3) ensure that current capacity is always nonnegative.

It should be noted here that P2 contains the flexibility of allowing $f_i < 0$ as might be the case if, instead of a salvage value for equipment, there were a positive cost of scrapping capacity in some period (in which case scrapping might be a means of avoiding unnecessary maintenance cost on unneeded equipment). For $f_i \geq 0$, P2 is bounded because of the initial assumptions made on c_i and f_i, and for $f_i < 0$,

P2 is bounded because of the form of the objective function.

As with P1, the objective function of P2 can be aggregated as follows:

$$\min Z_2 = \sum_{i=1}^{N}[p_i x_i + h_i v_i + (c_i + \sum_{j=i}^{N} m_j)w_i - (f_i + \sum_{j=i}^{N} m_j)r_j]$$
$$+ w_o \cdot \sum_{i=1}^{N} m_i$$

P1 becomes a special case of P2 if the scrapping cost is set to infinity (i.e., $f_i = -\infty \forall i$) in P2. Then $\bar{r} = 0$ always and constraints (P2.3) become redundant, giving back the form of P1. Two distinct models will be retained, however, for both economic and theoretical reasons.

Finally, in the CIDPI model, the value of fixed capital is central to the minimum cost solution. During periods 1 to N, this value is reflected in the salvage coefficient $\bar{f}$. To be complete, though, the model must include the value of fixed capital after period N, and here two cases are relevant; either the firm expects to go out of business after period N or it expects to continue production, but its planning horizon is limited by dependable demand projections. In both cases, a dummy period N + 1 can be added to the model

with $d_{N+1} = 0$. Then, in the former case, f_{N+1} is just the relevant salvage value of the fixed capital, while in the latter case, f_{N+1} can be set to equal some discounted estimate of the future value of production possible with the installed capacity.

As for the CPI model, nothing need be done since the salvage value of capital has no bearing on the problem. That is, capital only has value in the problem context to the extent that it can meet the demands.

I.5 Related Literature

A large literature exists in the area of production planning and there is at least a modest one on capacity expansion. But the combined problem, though economically significant, has been given relatively little attention, and no attention at all has been given to the necessity of providing good integer solutions to such a problem. A related literature closely allied with the combined problem does exist, however, in the area of employment, or work-force, smoothing. Each of these areas will be discussed subsequently.

Mention must first be made of the relationship among the existing capacity expansion literature

economics, and reality. Any demand-related problem naturally depends on a reasonable projection of demand, either stochastic or deterministic, within a believable horizon. Since forecasting is, at best, a risky business, the veracity of any demand projection must have an inverse relationship to the length of the horizon over which the projection is made. A reasonable figure for most companies would probably not exceed five years. Within such a time span, however, costs of production and inventory become economically significant, while beyond this span, any quantitative approach rapidly approaches speculation. The only valid conclusion then would seem to be that a realistic, quantitative approach to any capacity problem in the production sector must include the effects of the production and inventory policies.

A further economic and realistic condideration in the capacity expansion problem must be the fact that capacity, in whatever form it may take, is usually obtainable only in fixed increments. Furthermore, an incremental acquisition of capacity could, by the nature of the machine or whatever that is added, correspond to a large incremental capability in terms of units of output. Thus, although the roundoff of an exact, non-integer solution may give an answer that

is "close" to optimal in terms of the demand profile, the error in terms of cost could be very large. It must be concluded then that, at least in the context of capacity expansion, integer solutions are important.

Perhaps the classic early delineation of the capacity expansion problem was presented by Manne (Manne 61) in his analysis of the basic problem for both deterministic and stochastic demands. Early linear programming models of the problem were presented by Fetter (Fetter 61) and Veinott and Wagner (Veinott-Wagner 62). An equivalent form of the latters' model and a corresponding network formulation can easily be obtained from P2 by setting production and inventory costs to zero. Kalro and Arora (Kalro-Arora 70) extended the basic linear programming formulation to handle stochastic demands.

A dynamic programming approach to capacity expansion is given by Howard and Nemhauser (Howard-Nemhauser 68) for deterministic demands and extended to the stochastic demand case by Montgomery (Montgomery 71). An extension of the dynamic programming model for deterministic demands to handle concave costs of expansion is made by Manne and Veinott (Manne- Veinott 67). None of the above work takes into account the production-inventory problem, however, and thus will

not be considered as directly antecedent to this thesis.

Zabel (Zabel 63) was one of the early authors to look at the total problem, but his assumption of no inventory effectively reduces the problem he solves to one of capacity expansion. Hinomoto (Hinomoto 65) essentially retained Zabel's no-inventory assumption and extended his work to include technological improvement, thouch he couches the work in terms of replacement theory (failing to mention Zabel).

Iglehart (Iglehart 65) seems to have analyzed the first complete model for the CPI problem. He extends Zabel's work to allow inventory and restricted back-ordering using analytical techniques to establish optimal capital accumulation and production policies. He poses what seems to be a serious restriction, however, when he assumes that the consumptive factors of production are always spread evenly over all fixed capital, in effect eliminating the possibility of idle capacity. Carroll (Carroll 68) extended Iglehart's work to stochastic demands and pricing considerations using dynamic and non-linear programming techniques to establish optimal capital investment and production policies.

The most significant general formulation of the CPI problem to date, from a quantitative viewpoint,

has been the work of Cobian and Lele (Cobian-Lele 70). They give a general dynamic programming formulation of a multi-plant, multi-inventory facility firm faced with deterministic demands and show how the model can be hierarchically decomposed to simplify the solution process. Unfortunately, even when only one plant and one inventory facility are assumed, the model still contains two state variables and gives non-integer solutions.

The area of recent work which appears most relevant to the area of capacity expansion seems to have been done in the related literature on employment smoothing. Interpretationally, work force level can be viewed as a measure of capacity; the assumption made in this literature being that the fixed capacity level is given in each period, as opposed to assuming that the work force itself is optimally adjusted to the level of output, as mentioned in the previous section on economics. The greatest objection overcome in the employment smoothing literature is that, by nature of the problem, the production-inventory policy is always included. To date, however, integrality has not been considered an important aspect of the problem. An excellent delineation of the problem and methods of solution plus a bibliography through 1967 is given by

Silver (Silver 67).

A linear programming model which is close in form to P2 and a heuristic for "rounding-off" the linear programming solution are given by O'Malley, Elmaghraby, and Jeske (O'Malley et al. 66). As is well known, and demonstrated by example later in this thesis, rounding-off can be far from optimal and a very costly approximation to the optimal integer solution.

The major difference between P2 and the model of O'Malley et al. is the non-stationarity of all costs in the former, as opposed to the complete stationarity of costs in the latter. In addition, the capacity decrease coefficient in P2 can be positive or negative, within the bounds of the preconditions, to correspond to salvage value or scrapping cost respectively, while the corresponding term in the O'Malley model is a firing cost and constrained to be positive. Lastly, P2 is an integer program as opposed to the linear program of O'Malley. Still, O'Malley's model is not a special case of P2 since it has a term for overtime production. This, however, is a simple extension of P2 done in Chapter III, at which time the O'Malley model will become a special case (except for integrality).

The balance of the recent major work in

employment smoothing has been done in two companion papers by Lippman, Rolfe, Wagner, and Yuan (Lippman et al. 67a and 67b) and a later paper by Lippman and Yuan (Lippman-Yuan 69). The model addressed in the 1967 papers is a generalization of the linear model of O'Malley et al. with convex-like, non-stationary production costs, increasing and non-stationary inventory costs, and v-shaped, stationary smoothing costs. Demand-dependent upper and lower bounds on work force size are given and optimal policies are developed for the special cases of monotone increasing and monotone decreasing demands. Algorithms are then given for the special cases of monotonicity and the added restriction that all costs are linear, though non-stationarity is retained for production and inventory costs.

In the 1969 paper, similar analytical work is done on a model which is the same as that of 1967 with the exception that smoothing cost is permitted to be non-stationary. This paper also points out that the upper bounds and algorithms developed in the 1967 papers are invalidated under the more generalized smoothing costs. No further algorithms are presented.

The models in this thesis are special cases of the work of Lippman et al. only insofar as the more general form of the latter's production and inventory

cost functions are concerned. Integrality and the existence of algorithms in this work seem to make further comparison unnecessary. The most cogent point in the above discussion would seem to be the recognition of the interrelationship between capacity expansion literature and employment smoothing literature, about which more will be said in the final chapter.

I.6 Plan of the Work

In Chapter II, P1 and P2 will be transformed to the equivalent problems on which the algorithms given in the chapter are based. Bounds are developed here for the variable which acts as a surrogate for capacity. Finally, more efficient bounds and algorithms are presented for special monotone demand and cost cases.

Chapter III presents a network formulation of the entire problem for the special case when capacity increase and decrease costs are equal in each period, but are allowed to be non-stationary. The basic models are then extended to include indivisibility in acquisition of capacity and piecewise-linear, convex production costs that provide for consideration of multi-shift work and overtime.

A discussion of similar integer programming algorithms and a reinterpretation of the results of this thesis for the employment smoothing problem are presented in Chapter IV. Also presented here are the results of test problems run with a FORTRAN IV coding of the algorithm along with a summary and conclusions.

Chapter II

II.1 Introduction

Models P1 and P2 of Chapter I could be reformulated to eliminate the inventory variable v, in which case P1 would have 2N constraints and 2N variables and P2 would have 3N constraints and 3N variables. In the capacity expansion context with a period of one year and $N = 5$, neither problem would be very large and existing all integer algorithms could be used. On the other hand, in the employment smoothing context with a period of one month and $N = 24$, where furthermore the CIDPI model is most relevant, the problem size is out of the range of all except branch and bound type algorithms.

Use of implicit enumeration seems out of the question since the variables of P1 and P2 can take on rather large values, and expansion to 0 - 1 variables would quickly exceed the capacity of existing algorithms for even small values of N. Primal and dual cutting-plane algorithms have the disadvantages of highly unpredictable convergence and, for the latter, lack of a current feasible solution until optimality is reached. Although there is the possibility that one of these

other methods possesses an extremely efficient adaptation for this special problem, this thesis investigated the most obvious approach- branch and bound. In this chapter, special branch and bound algorithms will be developed for P1 and P2 which exploit the structure inherent in these problems.

In section II.2, a linear transformation is done on P1 to obtain an equivalent form P3 for which an algorithm can be more easily developed. The nature of the algorithm is discussed subsequent to the transformation. When treated as a function of certain of its variables, the optimal value of the objective function of P3 has convexity properties over certain ranges of those variables which are established in section II.3 by a series of lemmas, theorems and corollaries. The properties of the integer program P3 plus the results from section II.3 lead to the set of bounds and optimality tests given in section II.4 which are subsequently used in the branch and bound algorithm of section II.5.

A similar development is followed for the CIDPI model P2 in sections II.7 and II.8. The linear transformation used is the same and the resulting model P5 has properties similar to those of P3 both in problem structure and objective function convexity.

The modifications to the basic algorithm of section II.5 needed for P5 are given at the end of section II.8.

For both transformed models there are special cases of demand and cost structures which lead to major simplifications in the algorithms. These structures, their properties, and the attendent modifications of the algorithms are given in section II.6 for the CPI model and section II.9 for the CIDPI model.

II.2 Transformation of P1

A set of necessary and sufficient conditions for the feasibility of a $\bar{w}$ can be obtained from constraint sets (P1.1) and (P1.2). The method in general is to solve the inequalities (P1.1) for $\bar{x}$ by adding the appropriate terms to each side. Then for any $k = 1,2,\dots,N$, sum the first k constraints of (P1.2) to obtain a constraint of the form

$$x_1 + x_2 + \dots + x_k - v_k = \sum_{i=1}^{k} d_i - v_o$$

and substitute for the x_i from (P1.1) giving an inequality of the form

$$kw_1 + (k-1)w_2 + \dots + w_k - v_k \geq \sum_{i=1}^{k} d_i - v_o - kw_o$$

Repeating for each value of k gives the set

$$w_1 - v_1 \geq d_1 - v_o - w_o$$

(2.1)
$$\begin{aligned} 2w_1 + w_2 - v_2 &\geq d_1 + d_2 - v_o - 2w_o \\ &\vdots \\ Nw_1 + \ldots + w_N - v_N &\geq d_1 + \ldots + d_N - v_o - Nw_o \end{aligned}$$

If a vector of surplus variables $\bar{t} \geq 0$ is added to (2.1) and the v_i and t_i in each constraint of (2.1) are aggregated as

$$y_i = v_i + t_i \qquad i = 1,\ldots,N$$

then $\bar{y} \geq 0$ since $\bar{v} \geq 0$ and $\bar{t} \geq 0$, and (2.1) can be written in vector form as

(2.2)
$$B\bar{w} - \bar{y} = \hat{\bar{d}} - \bar{v}_o - \bar{w}_o$$

where

$$B = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 2 & 1 & 0 & \ldots & 0 & 0 \\ 3 & 2 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot & \cdot \\ \\ N & (N-1) & (N-2) & \ldots & 2 & 1 \end{bmatrix}$$

and the form of the vectors is obvious.

Solving for $\bar{w}$ from (2.2) gives

$$\bar{w} = B^{-1}\hat{\bar{d}} + B^{-1}\bar{y} - B^{-1}\bar{v}_o - B^{-1}\bar{w}_o$$

where

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ \\ 0 & 0 & 0 & 0 & . & . & . & 1 & -2 & 1 \end{bmatrix}$$

Expansion gives the following set of relations

$$\begin{aligned} w_1 &= d_1 + y_1 - v_o - w_o \\ w_2 &= d_2 - d_1 + y_2 - 2y_1 + v_o \\ w_3 &= d_3 - d_2 + y_3 - 2y_2 + y_1 \\ &\;\;\vdots \\ w_N &= d_N - d_{N-1} + y_N - 2y_{N-1} + y_{N-2} \end{aligned} \tag{2.3}$$

which expresses $\bar{w}$ explicitly in terms of demand, initial inventory, initial capacity, and the newly introduced aggregate variables $\bar{y}$. The structure of the right-hand side is rather unique and, as will be seen shortly, extremely useful. The above transformation will be referred to as Standard Transformation 1.

An interesting economic interpretation can be given to the variable y. If overcapacity is defined as the cumulative excess of capacity over cumulative demand up to and including period i, then y_i is the overcapacity in period i. For instance, if $N = 3$, $\bar{d}^T = (3, 5, 6)$, and $\bar{w}^T = (5, 0, 0)$, and $w_o = v_o = 0$ then $y_1 = y_2 = 2$ and $y_3 = 1$. Further, in any period i, v_i is the portion of overcapacity y_i which is used to produce for demand after period i and stocked as inventory, and t_i is the cumulative amount of idle overcapacity as of the end of period i. Thus, if $v_N = 0$, then y_N is the total idle capacity in the model.

Model P1 can be transformed by substituting for w_i from Standard Transformation 1 to give the following equivalent model.[1]

P3: $$\min Z_3 = \sum_{i=1}^{N} [p_i x_i + h_i v_i + c'_i y_i] + K$$

s.t.

(P3.1)
$$x_1 \leq d_1 + y_1 - v_o$$
$$x_i \leq d_i + y_i - y_{i-1} \qquad i = 2, \ldots, N$$

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[1] For the remainder of this thesis, the objective function of P_i will be referred to as Z_i.

$$x_1 - v_1 = d_1 - v_o$$

(P3.2) $$x_i + v_{i-1} - v_i = d_i \qquad i = 2,\dots,N-1$$

$$x_N + v_{N-1} = d_N + v_N$$

$$y_1 \geq w_o + v_o - d_1$$

(P3.3) $$y_2 - 2y_1 \geq d_1 - d_2 - v_o$$

$$y_i - 2y_{i-1} + y_{i-2} \geq d_{i-1} - d_i \qquad i = 3,\dots,N$$

$$\bar{x},\ \bar{v},\ \bar{y} \geq 0 \quad \text{and INTEGER}$$

where constraint set (P3.3) comes from substitution in $\bar{w} \geq 0$ in P1; the restriction of $\bar{y}$ to integers comes from the restriction of $\bar{w}$ to integers in P1 and the fact that the right-hand side of constraint set (P3.3) is integer by assumption, and where

$$c'_i = c_i - 2c_{i+1} + c_{i+2} + m_i - m_{i+1} \qquad i = 1,\dots,N-2$$

$$c'_{N-1} = c_{N-1} - 2c_N + m_{N-1} - m_N$$

$$c'_N = c_N + m_N$$

$$K = \sum_{i=1}^{N-1}[(c_i - c_{i+1} + m_i)\cdot d_i] + (c_N + m_N)d_N - c_1 w_o + (c_2 - c_1 - m_1)v_o$$

If, in P3, a feasible $\bar{y}$ is given then the problem

becomes

$$\text{P4:} \quad \min Z_4 = \sum_{i=1}^{N} [p_i x_i + h_i v_i]$$

s.t.

$$x_1 \leq d_1 + y_1 - v_o$$

$$\text{(P4.1)} \quad x_i \leq d_i + y_i - y_{i-1} \qquad i = 2,\ldots,N$$

$$x_1 - v_1 = d_1 - v_o$$

$$\text{(P4.2)} \quad x_i + v_{i-1} - v_i = d_i \qquad i = 2,\ldots,N-1$$

$$x_N + v_{N-1} = d_N + v_N$$

$$\bar{x}, \bar{v} \geq 0 \quad \text{and INTEGER}$$

which just is the production-inventory problem, parametric in $\bar{y}$; but which, more importantly, can be shown to be a transportation problem as was first pointed out by Bowman (Bowman 56). Specifically, let

x_{ij}- production in period i for demand in period j, $j \geq i$

then P4 can be written as

$$\text{P4':} \quad \min Z_4' = \sum_{i=1}^{N} \sum_{j=i}^{N} a_{ij} x_{ij}$$

s.t.

$$(P4.1') \qquad \sum_{j=i}^{N} x_{ij} \le \begin{cases} d_1 + y_1 - v_o & i = 1 \\ d_i + y_i - y_{i-1} & i = 2,\ldots,N \end{cases}$$

$$(P4.2') \qquad \sum_{i=1}^{j} x_{ij} = \begin{cases} d_1 - v_o & j = 1 \\ d_j & j = 2,\ldots,N-1 \\ d_N + v_N & j = N \end{cases}$$

$$\bar{x} \ge 0 \quad \text{and INTEGER}$$

where

$$a_{ij} = p_i + \sum_{k=i}^{j-1} h_k \qquad j = i+1,\ldots,N$$

$$a_{ii} = p_i$$

and $\bar{x}$ is integer by the properties of the transportation problem.[2]

The impact of this development is that, in a branch and bound algorithm enumeration would have to be done on only a subset of the problem variables, i.e. $\bar{y}$. For each feasible $\bar{y}$ generated, corresponding to a node in the enumeration, a transportation problem can be solved. This, in effect, implicitly fathoms each time what would have been a significant portion of the

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[2] The lower summation range in (P4.1') could be extended to 1 and the upper range in (P4.2') to N if $a_{ij} = \infty, \forall j < i,\ i = 2,\ldots,N$.

enumeration tree had branch and bound been tried on the entire problem. Furthermore, the number of transportation problems solved can be cut to a small fraction of the number of nodes actually visited by using suitable tests and bounds. In effect, the branch and bound is a search for the minimum cost combination of a feasible $\bar{y}$ and its corresponding optimal production-inventory schedule.

Lower bounds for $\bar{y}$ can be obtained directly from constraint set (P3.3) which can be solved for $\bar{y}$ as

$$\begin{aligned} & y_1 \geq w_o + v_o - d_1 \\ (2.4) \qquad & y_2 \geq 2y_1 + d_1 - d_2 - v_o \\ & y_i \geq 2y_{i-1} - y_{i-2} + d_{i-1} - d_i \qquad i = 3,\dots,N \end{aligned}$$

This set of lower bounds on $\bar{y}$ is integer since all terms on the right-hand side are integer.

Upper bounds on the problem variables must exist since the model is bounded . The following lemma will give a set of global upper bounds on $\bar{y}$.

Lemma 2.1 Global upper bounds for $\bar{y}$ are

$$(2.5) \qquad y_j \leq j \cdot \sum_{i=1}^{N} d_i - \sum_{i=1}^{j} d_i - (j-1)v_o \qquad j = 1,\dots,N$$

assumed that the remaining y_{j+k} are at their lower bounds for $k = 1,\ldots,N-j$.

Lemma 2.3 If for given y_i, $i = 1,\ldots,j$, the lower bounds of y_{j+k} are positive, $k = 1,\ldots,N-j$ then $w_{j+k} = 0$, $k = 1,\ldots,N-j$.

Proof: For any $k\varepsilon\{1,2,\ldots,N-j\}$ from (2.4) and the hypothesis

$$y_{j+k} \geq 2y_{j+k-1} - y_{j+k-2} + d_{j+k-1} - d_{j+k} > 0$$

Further, if y_{j+k} is at its lower bound, equality holds on the left. The expression for w_{j+k} from (2.3) is

$$w_{j+k} = y_{j+k} - 2y_{j+k-1} + y_{j+k-2} + d_{j+k-1} - d_{j+k}$$

but the right-hand side has just been shown to be zero. Thus

$$w_{j+k} = 0$$

QED

Lemma 2.4 If for given y_i, $i = 1,\ldots,j$, the lower bounds of y_{j+k} are positive, $k = 1,\ldots,N-j$, and $y'_j = y_j + n$, $n\varepsilon\{1,2,3,\ldots\}$ then

$$w'_j = w_j + n$$

and for $k = 1,2,\ldots,N-j$, the lower bound on y'_{j+k} is

$$y_{j+k} + (k+1)n$$

and

$$w'_{j+k} = 0$$

Proof: From (2.3)

$$w_j = y_j - 2y_{j-1} + y_{j-2} + d_j - d_{j-1}$$

so

$$w'_j = (y_j+n) - 2y_{j-1} + y_{j-2} + d_j - d_{j-1} = w_j + n$$

which proves the first assertion. The second and third are proved by induction on k. For $k = 1$, from (2.4)

$$y'_{j+1} \geq 2(y_j+n) - y_{j-1} + d_j - d_{j+1} = y_{j+1} + 2n$$

and from (2.3) and Lemma 2.3

$$\begin{aligned} w'_{j+1} &= (y_{j+1}+2n) - 2(y_j+n) + y_{j-1} + d_{j+1} - d_j \\ &= w_{j+1} = 0 \end{aligned}$$

Assume the relationship holds for a general $k \in \{2,3,\ldots, N-j-1\}$. Then for $k+1$ from (2.4)

$$y'_{j+k+1} \geq 2[y_{j+k}+(k+1)n] - [y_{j+k-1}+kn] + d_{j+k} - d_{j+k+1}$$
$$= y_{j+k+1} + (k+2)n$$

The final assertion is then true by Lemma 2.3.

QED

Remark: From the relationships (2.3) and (2.4) it can be seen that a change in y_j leaves $w_1,\ldots,w_{j-1}$ and $y_1, \ldots, y_{j-1}$ unaffected.

Now let

$C(y_j)$ = capacity cost as a function of y_j

$PI(y_j)$ = production-inventory cost as a function of y_j

$T(y_j)$ = total cost as a function of y_j

so

$$T(y_j) = C(y_j) + PI(y_j)$$

Then the preceeding lemmas can be used to establish the following cost properties for P3:

Theorem 2.1 If for given y_i, $i = 1,\ldots,j$, the lower bounds for y_{j+k} are positive $k = 1,\ldots,N-j$ then $C(y_j+n)$ is linear and non-decreasing in n for $n\varepsilon\{1,2,3\ldots\}$.

Proof: From Lemma 2.4 for $y'_j = y_j+n$, $n\varepsilon\{1,2,3\ldots\}$

$$w'_j = w_j + n$$
$$w'_{j+k} = 0 \qquad k = 1,2,\ldots,N-j$$

Thus, from P1

$$C(y_j+1) - C(y_j) = C(y_j+2) - C(y_j+1) = \ldots = (c_j + \sum_{i=j}^{N} m_i) \geq 0$$

QED

Before stating the next theorem, a well known result for the linear program associated with P4' must be mentioned. Since the linear program satisfies the sufficient conditions for unimodularity, then all extreme points of the convex hull are integer (Hu 69). This allows use of the properties of the general linear program in the following proof.

Theorem 2.2 If $n \epsilon \{1,2,3,\ldots\}$ and for any $j \epsilon \{1,2,\ldots,N\}$ if $y'_j = y_j + n$ and for each value of n the y_i are set at their lower bound, $i = j+1,\ldots,N$, then $PI(y'_j)$ is convex in n and the y_i, $i = j+1,\ldots,N$.

Proof: Assume the following linear programming problem

$$(p1): \quad \min \bar{c}^T\bar{x}$$

$$\text{s.t.} \quad A\bar{x} = \bar{b}$$

$$\bar{x} \geq 0$$

and the two parameterizations of $\bar{b}$

$$(p2) \qquad A\bar{x} = \bar{b} + (1-\lambda)\bar{\varepsilon}$$

$$(p3) \qquad A\bar{x} = \bar{b} + \bar{\varepsilon}$$

where $0 \leq \lambda \leq 1$, ε is any vector which maintains feasibility in (p3) and the optimal solution of (p1) is $\bar{x}_1$ and of (p3) is $\bar{x}_3$ where

$$c^T\bar{x}_1 = f(\bar{b})$$

$$c^T\bar{x}_3 = f(\bar{b}+\bar{\varepsilon})$$

are the optimal objective function values as functions of the right-hand sides. Now let

$$(2.6) \qquad \bar{x}_2' = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_3 \qquad 0 \leq \lambda \leq 1$$

and since the problems are linear it must be that

$$(2.7) \qquad \bar{c}^T\bar{x}_2 = \bar{c}^T\bar{x}_1 + (1-\lambda)\bar{c}^T\bar{x}_3$$

Multiplying the constraints of (p1) and (p3) by the appropriate constants gives

$$\begin{aligned} \lambda A\bar{x}_1 &= \lambda\bar{b} \\ (1-\lambda)A\bar{x}_3 &= (1-\lambda)(\bar{b}+\bar{\varepsilon}) \end{aligned} \qquad 0 \leq \lambda \leq 1$$

and since λ is a scalar, this can be rewritten as

$$A\lambda\bar{x}_1 = \lambda\bar{b}$$

$$0 \leq \lambda \leq 1$$

$$A(1-\lambda)\bar{x}_3 = (1-\lambda)(\bar{b}+\bar{\varepsilon})$$

Adding these two constraint sets gives

$$A[\lambda\bar{x}_1 + (1-\lambda)\bar{x}_3] = \bar{b} + (1-\lambda)\bar{\varepsilon}$$

which, from (2.6), is just

$$A\bar{x}_2' = \bar{b} + (1-\lambda)\bar{\varepsilon}$$

which proves that the convex combination of the optimal solutions to (p1) and (p3) is feasible to the convex combination of right-hand sides. Now if $\bar{x}_2$ is the optimal solution to (p2)

$$(2.8) \qquad \bar{c}^T\bar{x}_2 \leq \bar{c}^T\bar{x}_2'$$

since it is a minimization problem.

From (2.7) and (2.8)

$$\bar{c}^T\bar{x}_2 \leq \lambda\bar{c}^T\bar{x}_1 + (1-\lambda)\bar{c}^T\bar{x}_3 \qquad 0 \leq \lambda \leq 1$$

producing the general convexity result and, by the remark preceding this theorem, the desired result for $PI(y_j+n)$.

QED

The following corollaries follow trivially.

Corollary 2.2.1 $PI(y_N+n)$ is convex for $n \in \{1,2,3,\ldots\}$.

Corollary 2.2.2 If for given y_i, $i = 1,\ldots,j$, the lower bounds of y_{j+k} are positive, $k = 1,\ldots,N-j$ then $PI(y_j+n)$ is convex in n for $n \in \{1,2,3,\ldots\}$.

Proof: From Lemma 2.4 the capacities in periods j+1 through N are equal to the capacity in period j. The result follows from this and Theorem 2.2. QED

The major result can now be given as follows:

Theorem 2.3 If for given y_i, $i = 1,\ldots,j$, the lower bounds for y_{j+k} are positive, $k = 1,\ldots,N-j$, then $T(y_j+n)$ is convex in n for $n \in \{1,2,3,\ldots\}$.

Proof: The proof follows from Theorem 2.1 and Corollary 2.2.2 and the fact that the sum of convex functions is convex. QED

Corollary 2.3.1 $T(y_N+n)$ is convex for $n \in \{1,2,3,\ldots\}$.

II.4 Optimality Tests and Bounds for the Enumeration

The results of section II.3 can now be used to develop bounds and optimality tests for the

algorithm. The logical first step is to solve the linear programming problem for P1 and use

Optimality Test 1: If the optimal linear programming solution is all integer, then this is an optimal solution to P3.

Otherwise let

Z^*_{LP} = optimal objective function value of the linear program associated with the integer programming problem

Z^*_{IP} = optimal integer program objective function value

It is a well known fact that

$$Z^*_{LP} \leq Z^*_{IP}$$

and this is used as

Bound 1: A lower bound on total cost is Z^*_{LP}.

The algorithm will use Bound 1 in the following optimality test.

Optimality Test 2: If Z_{IP} is the current integer programming solution and

$$Z_{IP} = \lceil Z^*_{LP} \rceil$$

then Z_{IP} is an optimal solution to the integer

program.[3]

The next bound uses the minimum production-inventory cost, PI*, to obtain an upper bound on capacity expansion cost. PI* can be obtained by solving P4' with the values of the right-hand sides of constraint set (P4.1') equal to infinity. This corresponds to putting an infinite supply at the sources of the transportation problem.

Lemma 2.8 If Z_3' is a current optimal solution in the enumeration then

$$\hat{\alpha} = Z_3' - PI^*$$

is an upper bound on capacity expansion cost.

Proof: Assume there is a capacity expansion cost α^1 such that

$$\alpha^1 > \hat{\alpha}$$

and with a corresponding Z_3^1 such that

$$Z_3^1 < Z_3'$$

But this implies that

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[3] $\lceil a \rceil$ indicates the next integer greater than a.

$$PI = Z_3^1 - \alpha^1 < Z_3' - \hat{\alpha} = PI^*$$

which is a contradiction of the construction of PI*.

QED

Using the above gives:

Bound 2: If Z_3' is a current optimal solution of the integer program, then an upper bound on capacity expansion cost α is

$$\alpha \leq Z_3' - PI^* = \hat{\alpha}$$

Lemma 2.8 can be useful in another way. If the current upper bound on capacity cost is $\hat{\alpha}$ then certainly an upper bound on period 1 capacity is[4]

$$w_1 \leq \left\lfloor \frac{\hat{\alpha}}{c_1 + \sum_{i=1}^{N} m_i} \right\rfloor$$

since any larger value of w_1 would violate $\hat{\alpha}$. A similar bound for w_2 is

$$w_2 \leq \left\lfloor \frac{\hat{\alpha} - (c_1 + \sum_{i=1}^{N} m_i) w_1}{(c_2 + \sum_{i=2}^{N} m_i)} \right\rfloor$$

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[4] $\lfloor a \rfloor$ indicates the greatest integer smaller than a.

In general the following bound can be used.

Bound 3: For the current $\hat{\alpha}$, upper bounds on capacity expansion in each period are

$$w_1 \le \left\lfloor \frac{\hat{\alpha}}{c_1 + \sum_{i=1}^{N} m_i} \right\rfloor$$

$$w_j \le \left\lfloor \frac{\hat{\alpha} - \sum_{i=1}^{j-1} \left(c_i + \sum_{k=i}^{N} m_k\right) w_i}{c_j + \sum_{k=j}^{N} m_k} \right\rfloor \qquad j = 2,\ldots,N$$

The following bounds use the results of the previous sections.

Bound 4: For any feasible values of y_i, $i = 1,\ldots,j$, lower bounds on the values of y_i, $i = j+1,\ldots,N$ can be obtained from (2.4).

Bound 5: Global upper bounds on y_i, $i = 1,\ldots,N$ can be obtained from (2.5).

Optimality Test 3: If enumeration has been completed with y_1 at its global upper bound or the upper bound inferred from Bound 3, whichever is lower, then the current optimal solution is optimal for the integer program.

The following bound on y_N uses the results of Corollary 2.3.1.

Bound 6: If for a given set of y_i, $i = 1,\ldots,N-1$

$$T(y_N) \leq T(y_N+1)$$

then the enumeration on y_N for this set of y_i can be fathomed at (y_N+1).

The following bound uses the result of Theorem 2.3.

Bound 7: If for a given $y_1, y_2,\ldots,y_j$, the lower bounds on y_i are positive for $i = j+1,\ldots,N$ and if

$$T(y_j) \leq T(y_j+1)$$

then the enumeration on y_j can be fathomed for this set of y_i, $i = 1,\ldots,j-1$ at (y_j+1).

Bounds 8 and 9 make use of Theorem 2.1 and Corollary 2.1.1.

Bound 8: For given y_i, $i = 1,\ldots,j$, if the lower bounds on y_i are positive for $i = j+1,\ldots,N$ and if $C(y_j) \geq \hat{\alpha}$, then y_j is fathomed for the preceding set of y_i.

Bound 9: If for given y_i, $i = 1,\ldots,N-1$, $C(y_N) \geq \hat{\alpha}$, then no further enumeration on y_N is necessary.

Since the algorithm conducts the search by enumeration on y_N most often, Bounds 6 and 9 become very powerful tools. Bounds 7 and 8 are also extremely powerful because their conditions will be satisfied

often during enumeration on the later period variables due to relationships (2.4). The usefulness of these bounds diminishes the need for the global upper bounds (2.5) since such bounds must include all cases where Bounds 6 through 9 would be useful.

II.5 Algorithm for P3

Step 0 Initialization Set the value of current optimal cost, Z_3', and the current bound on capacity cost, $\hat{\alpha}$, to infinity.

Step 1 Solve the associated linear program. If the solution is integer, stop. (Optimality Test 1) Otherwise set the linear programming bound to (Bound 1)

$$\lceil Z^*_{LP} \rceil$$

and go to Step 2a.

Step 2a Solve an uncapacitated transportation problem to get the minimum production-inventory cost PI* and go to Step 2b.

Step 2b Find global upper bounds for y_i, $i = 1,\ldots,N$ from (2.5). Go to Step 3.

Step 3 Start the enumeration by setting

$$y_1 = \max\,[(w_o+v_o-d_1),0]$$

and finding lower bounds for the remaining values of y_i, $i = 2,\ldots,N$, from (2.4), using this value of y_1. The pointer (PNT) is N. Go to Step 5.

Step 4a PNT = PNT - 1. Go to Step 4b.

Step 4b With PNT = j, let $y_j = y_j+1$ and, if $j < N$, find new lower bounds for succeeding y_i from (2.4). If $j \neq 1$ and any lower bound exceeds an upper bound (from Bounds 3 and 5), y_j is fathomed so go to Step 4a with PNT = j. If $j = 1$ and some lower bound exceeds an upper bound, go to Step 10 (Optimality Test 3). Otherwise go to Step 5 with PNT = j.

Step 5 With PNT = j determine capacity cost $\alpha = C(y_j)$. If $\alpha > \hat{\alpha}$ and either $j = N$, or $j < N$ and $y_i > 0$, $i = j+1, \ldots, N$ go to Step 4a with PNT = j (Bounds 8 and 9). Otherwise if $\alpha > \hat{\alpha}$ go to Step 4b with PNT = N. If $\alpha \leq \hat{\alpha}$ and if $j < N$ with $y_i > 0$, $i = j+1,\ldots,N$ go to Step 6 with PNT = j (This sets up convexity test for Bound 7). Otherwise go to Step 6 with PNT = N (This sets up convexity test for Bound 6).

Step 6 With PNT = j solve a transportation problem to obtain the related optimal production-inventory cost $PI(y_j)$. Then total cost is $T(y_j) = C(y_j) + PI(y_j)$. Go to Step 7 with PNT = j.

Step 7 With PNT = j, if $j = N$ or $j < N$ and $y_i > 0$, $i = j+1,\dots,N$ and $T(y_j) > T(y_j - 1)$ then y_j is fathomed so go to Step 4a with PNT = j (Bounds 6 and 7). If $T(y_j) = T(y_j - 1) = Z'_3$ in any case go to Step 9. For all other cases if $T(y_j) < Z'_3$ go to Step 8, otherwise go to Step 4b with PNT = N.

Step 8 If $T(y_j) = \lceil Z^*_{LP} \rceil$ go to Step 10 (Optimality Test 2). Otherwise go to Step 9.

Step 9 Either include $T(y_j) = Z'_3$ on the list of current optimal solutions or replace the current list with $Z'_3 = T(y_j)$, whichever is appropriate. Update intermediate bounds from Bound 3 and go to Step 4b with PNT = N.

Step 10 Decode the current optimal solutions and STOP.

The algorithm, as written, will also find all alternative optimal solutions if termination is not due to Step 8. The flexibility of choice among solutions would normally be desirable in a management situation. If the alternate optima and longer running times are not desired then the following changes to the algorithm are needed:

Step 5 With PNT = j determine capacity cost $\alpha = C(y_j)$. If $\alpha \geq \hat{\alpha}$ and either $j = N$ or $j < N$ and $y_i > 0$, $i = j+1, \dots, N$, go to Step 4a with PNT = j. Otherwise if $\alpha \geq \hat{\alpha}$ go to Step 4b with PNT = N. If $\alpha < \hat{\alpha}$ and if $j < N$ with

$y_i > 0$, $i = j+1,\ldots,N$ go to Step 6 with PNT = j. Otherwise go to Step 6 with PNT = N.

Step 7 With PNT = j, if $j = N$ or $j < N$ and $y_i > 0$, $i = j+1,\ldots,N$ and $T(y_j) \geq T(y_j-1)$ then y_j is fathomed so go to Step 4a with PNT = j. If $T(y_j) < Z_3'$ go to Step 8, otherwise go to Step 4b with PNT = N.

Step 9 Replace the current optimal solution with $Z_3' = T(y_j)$. Go to Step 4b with PNT = N.

II.6 The CPI Model Under Special Demand and Cost Assumptions

In this section two special cases of the CPI model will be discussed, one involving monotone non-decreasing demands and the other monotone non-increasing demands. The former leads to significant simplifications in the CPI algorithm, while the latter is a trivial case of the CPI model.

The following two cost assumptions hold in this section:

(a) The cost assumptions of P1 hold.

(b) The cost coefficients of P1 (Chapter I) are monotone non-increasing with time.

Assumption (b) allows the possibility of discounting.

Monotone non-decreasing demand

The case of demand which is monotone non-decreasing is likely to occur often in the context of firms interested in solving capacity expansion problems. The main result for this case is contained in the following theorem.

Theorem 2.4 If assumptions (a) and (b) hold, demand is monotone non-decreasing, and

$$Nw_o + v_o \leq \sum_{i=1}^{N} d_i$$

then there is an optimal solution with $y_N = 0$.

Proof: Assume in contradiction that

$$y_N = a > 0$$

It can also be assumed, as discussed in Chapter I, that $v_N = 0$. But this just implies that there are $a > 0$ idle units of capacity in the model. Consider the first unit of idle capacity. One of two cases can be assumed to exist.

Case I- The idle unit was added in period j instead of k where it was first used, where $j < k \leq N$.

Case II- The capacity was added in some period previous

to j, was used up to and including j-1, and became idle in j.

For Case I to happen either the cost of adding capacity in j must be cheaper than k, i.e.

$$c_j + \sum_{i=j}^{N} m_i < c_k + \sum_{i=k}^{N} m_i$$

which, since $c_j \geq c_k$ (assumption (b)) implies that $\sum_{i=j}^{k-1} m_i < 0$ contradicting assumption (a); or the cost is equal to that in k in which case the idle unit could be removed in j and added in k with no change in cost.

For Case II it must be that $w_j = 0$ since if not, then Case I applies. So this unit of capacity must have been built in some previous period q and used to produce from q through j-1. Now at least one of these (j-q) units of production must be inventoried into period j since if $v_{j-1} = 0$ then since $x_{j-1} > x_j$ this implies that $d_{j-1} > d_j$ contradicting the demand assumption. Furthermore, by the same argument, $v_i \geq 1$, $i = q, q+1, \ldots, j-1$ so the idle unit in j was used to produce for inventory in period q and at least that unit of inventory was carried into period j. The production schedule of periods q+1 to j-1 will not change if the unit of capacity is added in q+1 instead of q, however the unit which was inventoried from q to j could be

produced in j instead, eliminating the idle unit of capacity. With idle capacity the incremental cost considering just the one unit of production in question was

$$(2.9) \qquad c_q + \sum_{i=q}^{N} m_i + p_q + \sum_{i=q}^{j-1} h_i$$

while adding the capacity in q+1 and producing in j gives

$$(2.10) \qquad c_{q+1} + \sum_{i=q+1}^{N} m_i + p_j$$

Subtracting (2.10) from (2.9) gives

$$(c_q - c_{q+1}) + m_q + (p_q - p_j) + \sum_{i=q}^{j-1} h_i \geq 0$$

by assumptions (a) and (b). Thus, in both cases, idle capacity can be reduced with, at worst, no reduction in cost.

If idle capacity still exists, the argument can be repeated.

QED

Corollary 2.4.1: If the assumptions of Theorem 2.4 hold then all capacity is used in each period.

The use of Corollary 2.4.1 leads to an imme-

diate simplification of the production-inventory problem P4. Since the right-hand sides of constraint set (P4.1) are just capacities, equality holds in all N constraints and the x_i are determined trivially for a given $\bar{y}$. Constraint set (P4.2) becomes N equations in N unknowns (since v_o is explicit and v_N is either zero implicitly or set explicitly) and is thus deterministic. So the subproblem in the enumeration can be solved by inspection each time.

Another observation follows from Theorem 2.4 and Lemma 2.4.1. If all capacity is always used it can certainly never be feasible to build capacity in any period sufficient to produce all remaining demand. In fact, the upper bounds on capacity increase are much smaller than those given by (2.5). Instead of stating them explicitly, they can be handled implicitly in the algorithm by the following:

Bound 10: For given y_i, $i = 1,\ldots,j$, if the increment

$$y_j' = y_j + 1$$

causes the succeeding lower bounds to contain $y_N > 0$, then y_j is fathomed for the preceding set of y_i.

Bounds 6 through 9 are now useless, since their conditions are never met in this case, so

Bound 10 takes over as a main fathoming rule in the algorithm. Furthermore, the global upper bounds from (2.5) are irrelevant to this special case and can be ignored.

The basic algorithm requires the following modifications:

1) Drop Step 2b.

2) Step 4b' With PNT = j let $y_j = y_j+1$ and find new lower bounds for the succeeding elements from (2.4). If $y_N > 0$ then y_j is fathomed so go to Step 4a with PNT = j (Bound 10). If j = 1 and y_j exceeds its upper bound from Bound 3 or $y_N > 0$, go to Step 10. Otherwise go to Step 5' with PNT = j.

3) Step 5' With PNT = j determine capacity cost $\alpha = C(y_j)$. If $\alpha > \hat{\alpha}$ go to Step 4b' with PNT = N-1. Otherwise go to Step 6' with PNT = j.

4) Step 6' With PNT = j, determine the related optimal production-inventory schedule by inspection and the related cost $PI(y_j)$. Total cost is $T(y_j) = C(y_j) + PI(y_j)$. Go to Step 7' with PNT = j.

5) Step 7' With PNT = j if $T(y_j) > Z'_3$ go to Step 4b' with PNT = N-1. If $T(y_j) = Z'_3$ go to Step 9'. Otherwise go to Step 8.

6) Step 9' Either include $T(y_j) = Z'_3$ on the list of current optimal solutions or replace the current list

with $Z'_3 = T(y_j)$, whichever is appropriate. Go to Step 4b' with PNT = N-1.

Monotone non-increasing demand

Assume that demand is monotone non-increasing and that monotonicity is not destroyed by initial inventory, i.e.

$$d_1 - v_o \geq d_2$$

By assumption (b) production should always take place as late as possible. Then for the CPI model the only optimal capacity expansion policy can be to expand capacity in the first period to satisfy the demand in that period, i.e.

$$w_1 = d_1 - v_o - w_o$$

This will then be sufficient to produce the remaining demands, so no capacity expansion problem need be solved. If the cost assumptions do not hold however, then this case is just handled with the general CPI model.

II.7 Transformation for P2

A similar procedure to that used in transforming the CPI model P1 will be used to transform the CIDPI model P2. First a lemma will be proved which will be useful in interpreting the transformed model, and in the algorithm to follow.

Lemma 2.9 For any period i either

(1) $w_i > 0$ and $r_i = 0$

or

(2) $w_i = 0$ and $r_i = 0$

or

(3) $w_i = 0$ and $r_i > 0$

Proof: The proof is by contradiction and construction. Assume that $w_i > 0$ and $r_i > 0$ for any period i gives a minimum cost. There are three cases:

Case 1 $w_i - r_i > 0$

Let $w'_i = w_i - r_i$ and $r'_i = 0$

Then

$$\begin{aligned} c_i w'_i - f_i r'_i + (w'_i - r'_i)\cdot\sum_{j=i}^{N} m_j &= c_i(w_i - r_i) - f_i \cdot 0 + (w_i - r_i)\cdot\sum_{j=i}^{N} m_j \\ &= c_i w_i - c_i r_i + (w_i - r_i)\cdot\sum_{j=i}^{N} m_j \\ &< c_i w_i - f_i r_i + (w_i - r_i)\cdot\sum_{j=i}^{N} m_j \end{aligned}$$

since $c_i > f_i$.

The assumption is contradicted and (1) has been constructed.

Case 2 $w_i - r_i = 0$

Let $w'_i = 0$ and $r'_i = 0$

Then certainly

$$0 < c_i w_i - f_i r_i$$

since $c_i > f_i$

The assumption is again contradicted and (2) has been constructed.

Case 3 $w_i - r_i < 0$

Let $w'_i = 0$ and $r'_i = r_i - w_i$

Then

$$\begin{aligned} c_i w'_i - f_i r'_i + (w_i - r_i) \cdot \sum_{j=1}^{N} m_j &= c_i \cdot 0 - f_i (r_i - w_i) + (w_i - r_i) \cdot \sum_{j=1}^{N} m_j \\ &= f_i w_i - f_i r_i + (w_i - r_i) \cdot \sum_{j=1}^{N} m_j \\ &< c_i w_i - f_i r_i + (w_i - r_i) \cdot \sum_{j=1}^{N} m_j \end{aligned}$$

since $c_i > f_i$

The contradiction is complete and (3) has been constructed.

QED

Now using Standard Transformation 1 as in Section II.2, constraint sets (P2.1) and (P2.2) can be combined to give the set of relations

$$
(2.11)\qquad
\begin{aligned}
(w_1-r_1) &= d_1 + y_1 - w_o - v_o \\
(w_2-r_2) &= d_2 - d_1 + y_2 - 2y_1 + v_o \\
(w_3-r_3) &= d_3 - d_2 + y_3 - 2y_2 + y_1 \\
&\ \ \vdots \\
(w_N-r_N) &= d_N - d_{N-1} + y_N - 2y_{N-1} + y_{N-2}
\end{aligned}
$$

Note here that, by Lemma 2.9 and $\bar{w}$, $\bar{r} \geq 0$, if the right-hand side of a constraint in (2.11) is positive, then $w_i > 0$ and $r_i = 0$. If it is zero, then $w_i = r_i = 0$. If it is negative, then $w_i = 0$ and $r_i > 0$.

Solving (2.11) for $\bar{r}$ and substituting in (P2) gives

$$
\text{P5:}\qquad \min Z_5 = \sum_{i=1}^{N} [p_i x_i + h_i v_i + f'_i y_i + (c_i - f_i) w_i] + M
$$

s.t.

$$
(\text{P5.1})\qquad
\begin{aligned}
x_1 &\leq d_1 + y_1 - v_o \\
x_i &\leq d_i + y_i - y_{i-1} \qquad i = 2, \ldots, N
\end{aligned}
$$

$$x_1 - v_1 = d_1 - v_o$$

(P5.2) $$x_i + v_{i-1} - v_i = d_i \qquad i = 2,\ldots,N-1$$

$$x_N + v_{N-1} = d_N + v_N$$

$$d_1 + y_1 - v_o \geq 0$$

(P5.3) $$d_i + y_i - y_{i-1} \geq 0 \qquad i = 2,\ldots,N$$

$$w_1 \geq d_1 + y_1 - (w_o + v_o)$$

(P5.4) $$w_2 \geq d_2 - d_1 + y_2 - 2y_1 + v_o$$

$$w_i \geq d_i - d_{i-1} + y_i - 2y_{i-1} + y_{i-2} \qquad i = 3,\ldots,N$$

$$\bar{x},\ \bar{v},\ \bar{w},\ \bar{y} \geq 0 \quad \text{and INTEGER}$$

where

$$f'_i = f_i - 2f_{i+1} + f_{i+2} + m_i - m_{i+1} \qquad i = 1,\ldots,N-2$$

$$f'_{N-1} = f_{N-1} - 2f_N + m_{N-1} - m_N$$

$$f'_N = f_N + m_N$$

$$M = \sum_{i=1}^{N-1}(f_i - f_{i+1})d_i + f_N d_N + \sum_{i=1}^{N} m_i d_i - f_1 w_o + (f_2 - f_1 - m_1)v_o$$

Constraint sets (P5.1) and (P5.2) and the

first two sets of terms in the objective function are just problem P4 and thus are equivalent to a transportation problem. Writing constraint set (P5.4) separately with the last set of terms of the objective function gives

$$\text{P6:} \qquad \min Z_6 = \sum_{i=1}^{N} (c_i - f_i)\, w_i$$

s.t.

$$w_1 \geq d_1 + y_1 - w_o$$

$$w_N \geq d_N - d_{N-1} + y_N - 2y_{N-1} + y_{N-2}$$

$$w \geq 0 \text{ and INTEGER}$$

which, given $\bar{y}$, is solvable for $\bar{w}$ by inspection since the coefficients in the objective function are positive and the right-hand sides of the constraints are integer. Furthermore, it is possible that the right-hand side of some constraint i is negative or zero. Recalling Lemma 2.9, it follows that in this case $w_i = 0$ and r_i equals the absolute value of the right-hand side. So, for each possible $\bar{y}$ in the CIDPI model, it is necessary to consider a transportation problem and a trivial linear program.

Essentially the same branch and bound enumeration scheme can be followed here as was used in the capacity increase case. Lemma 2.1 still applies as a finite bound on the enumeration and (2.5) is still a valid set of global upper bounds.

The lower bounds on $\bar{y}$ are of a slightly different form than (2.4) and can be gotten from (P5.3) and $\bar{y} \geq 0$ as

$$\begin{aligned} & y_1 \geq v_o - d_1 \\ (2.12) \qquad & y_i \geq y_{i-1} - d_i \qquad i = 2,\ldots,N \\ & \bar{y} \geq 0 \end{aligned}$$

II.8 Properties and Algorithm for P5

The properties of P5 are similar to those of P3. Because of the new lower bounds (2.12) however, most of the results must be restated and reproved.

The statement and proof of Lemma 2.2 hold equivalently for P5 if the change of variables

$$w_i = (w_i - r_i) \qquad i = 1,\ldots,N$$

is made. To arrive at sufficient conditions for convexity to hold in the optimal values of Z_5, the remaining results must be reiterated. For Lemmas 2.10, 2.11 and 2.12 and Theorems 2.5 and

and 2.6, when given a set of y_i, $i = 1,\ldots,j$, it is assumed that the remaining y_{j+k} are at their lower bounds for $k=1,\ldots,N-j$. Lemma 2.3 is restated as

Lemma 2.10 If for given y_i, $i = 1,\ldots,j$ the lower bounds of y_{j+k} are positive, $k = 1,\ldots, N-j$ then $(w_{j+k}-r_{j+k}) = 0$ for $k = 2,\ldots, N-j$.

Proof: For any $k\varepsilon\{2,3,\ldots,N-j\}$ from (2.12)

$$y_{j+k} \geq y_{j+k-1} - d_{j+k} > 0$$

and

$$y_{j+k-1} \geq y_{j+k-2} - d_{j+k-1} > 0$$

where equality holds on the left. Then from (2.11)

$$\begin{aligned}(w_{j+k}-r_{j+k}) &= (y_{j+k}-y_{j+k+1}+d_{j+k})-(y_{j+k-1}-y_{j+k-2}+d_{j+k-1})\\ &= 0\end{aligned}$$

QED

Next, Lemma 2.4 becomes

Lemma 2.11 If the conditions of Lemma 2.10 hold and $y_j' = y_j+n$ for $n\varepsilon\{1,2,3,\ldots\}$ then

$$(w_j-r_j)' = (w_j-r_j) + n$$

and for $k = 1,2,\ldots,N-j$ the lower bound on y_{j+k}' is

$$y_{j+k} + n$$

and

$$(w_{j+1}-r_{j+1})' = (w_{j+1}-r_{j+1}) - n$$

and

$$(w_{j+k}-r_{j+k})' = 0 \qquad k = 2,\ldots,N-j$$

Proof: From (2.11)

$$(w_j-r_j)' = d_j - d_{j-1} + (y_j+n) - 2y_{j-1} + y_{j-2}$$
$$= (w_j-r_j) + n$$

For $k = 1$

$$y'_{j+1} \geq y'_j - d_{j+1} = (y_j+n) - d_{j+1} = y_{j+1} + n$$

so from (2.11)

$$(w_{j+1}-r_{j+1})' = d_{j+1} - d_j + (y_{j+1}+n) - 2(y_j+n) + y_{j-1}$$
$$= (w_{j+1}-r_{j+1}) - n$$

Doing induction on k, for $k = 2$

$$y'_{j+2} \geq y'_{j+1} - d_{j+2}$$

$$= (y_{j+1}+n) - d_{j+2}$$

$$= y_{j+2} + n$$

Assume the relationship holds for a general k.

Then for k+1

$$y'_{j+k+1} \geq y'_{j+k} - d_{j+k+1}$$

$$= (y_{j+k}+n) - d_{j+k+1}$$

$$= y_{j+k+1} + n$$

Finally by Lemma 2.10

$$(w_{j+k}-r_{j+k})' = 0 \qquad k = 2,\ldots,N$$

QED

The following is an additional property which will be needed.

Lemma 2.12 If the conditions of Lemma 2.10 hold, then $(w_{j+1}-r_{j+1})' \leq 0$.

Proof: From (2.11)

$$(w_{j+1}-r_{j+1})' = (y'_{j+1}-y'_j+d_{j+1}) - (y'_j-y_{j-1}+d_j) \leq 0$$

since y'_{j+1} is at its lower bound so the first term is zero, and the second term must be non-negative by (2.12)

QED

Now Theorem 2.1 can be restated as

Theorem 2.5 If for given y_i, $i = 1,\dots,j$, the lower bounds on y_{j+k} are positive for $k = 1,\dots,N-j$, then $C(y_j+n)$ is strictly convex in n for $n\varepsilon\{1,2,3,\dots\}$.

Proof: For $y_j' = y_j+n$, $n\varepsilon\{1,2,3,\dots\}$from Lemmas 2.11 and 2.12

$$(w_j-r_j)' = (w_j-r_j) + n$$

$$(w_{j+1}-r_{j+1})' = (w_{j+1}-r_{j+1}) - n \leq 0$$

$$(w_{j+k}-r_{j+k})' = 0 \qquad\qquad k = 2,\dots,N$$

The capacity change variables in period j could start from $r_j > 0$ and go to $w_j > 0$ as n increases. Assume there is an n_1 such that $r_j > 0$ for $n < n_1$ and $w_j > 0$ for $n > n_1$. Then the change in capacity cost for each increment of $n < n_1$ is

$$\Delta_1 = f_j + m_j - f_{j+1}$$

while for $n \geq n_1$ the cost change is

$$\Delta_2 = c_j + m_j - f_{j+1} > 0$$

where $\Delta_2 > 0$ follows from the cost assumption of P2

that

$$c_i + \sum_{k=i}^{N} m_k > f_j + \sum_{k=j}^{N} m_k \qquad \forall j > i,\ i = 1,\ldots,N-1$$

Further from P2 it was assumed that $c_j > f_j$ giving the convexity result

$$\Delta_1 < \Delta_2$$

QED

Corollary 2.5.1 $C(y_N+n)$ is convex in n for $n\varepsilon\{1,2,3,\ldots\}$.

Theorem 2.2 and Corollary 2.2.1 still hold since, as has been pointed out, the production-inventory section of the model is equivalent. The main result, Theorem 2.3, must be restated as

Theorem 2.6 If the conditions of Theorem 2.5 hold, then $T(y_j+n)$ is convex in n for $n\varepsilon\{1,2,3,\ldots\}$.

Proof: The proof follows from Theorem 2.5 and Corollary 2.2.2 and the fact that the sum of convex functions is convex.

QED

Corollary 2.3.1 is unchanged. The above properties of the P5 variables and convexity in the optimal values of Z_5 will, with a few exceptions, permit the bounds and optimality tests of section II.4 to be used.

Algorithm for the CIDPI model

The bounds and optimality tests of section II.4 hold for the CIDPI model with a few exceptions. Bound 3 is no longer valid since the possibility of decreasing capacity allows a w_i to exceed these bounds and not violate $\hat{\alpha}$. For Bound 4 the lower bounds are now found from (2.12). Optimality Test 3 can no longer use Bound 3. Finally, since Theorem 2.5 only gives the weaker result of convexity for $C(\bar{y})$, Bound 8 has to be rewritten to ensure that the test is being conducted in the area where the slope of $C(\bar{y})$ is positive.

Bound 8' For given y_i, $i = 1,\ldots,j$ if $y_i > 0, i = j+1, \ldots, N$, $C(y_j) \leq C(y_j+1)$, and $C(y_j+1) \geq \hat{\alpha}$ then y_j is fathomed for the preceding set of y_i.

The basic algorithm of section II.4 requires some changes for the CIDPI model. The lower bounds in Steps 3 and 4b are now found from (2.12) and the initial value for y_1 in Step 3 is

$$y_1 = \max\ [(v_0-d_1),\ 0]$$

The first part of Step 5 must be rewritten to reflect Bound 8' as follows: With PNT = j determine capacity cost $\alpha = C(y_j)$. If $\alpha > \hat{\alpha}$ and either $j = N$ or $j > N$,

$y_i > 0$ for $i = j+1,\ldots,N$ with $C(y_j) \geq C(y_j-1)$ go to Step 4a with PNT = j (Bounds 8' and 9)..... The remainder of Step 5 is the same. Finally, the update of Bound 3 in Step 9 can be dropped.

The remarks of section II.5 pertaining to alternate optimal solutions are relevant here also. Finally, note that lower bounds (2.12) will serve to increase the feasible space for $\bar{y}$ and thus longer average running times should result for the CIDPI model.

II.9 Cases of Monotone Demands with Special Cost Structures for the CIDPI Model

As with the CPI model, the efficiency of the algorithm for the CIDPI model is increased greatly when monotonicity exists in demand. The following hold for this section

a) The cost coefficients of P2 (Chapter I) are monotone non-increasing with time.

b) The cost assumptions of P2 hold.

Monotone non-decreasing demand case

Lemma 2.13 If assumptions (a) and (b) of this section hold, demand is monotone non-decreasing and

$$Nw_o + v_o \leq \sum_{i=1}^{N} d_i$$

then there is an optimal solution with $y_N = 0$.

Proof: The proof follows trivially from the fact that these are the same conditions as for Theorem 2.4. Now, in addition to the possibility of eliminating idle capacity by the arguments of Theorem 2.4, there is the added possibility of scrapping it and recovering a salvage value, so the results hold a fortiori. QED

The conclusions of section II.6 which follow from Theorem 2.4 are valid here for the result above; i.e. the transportation subproblem becomes trivial and can be solved by inspection for a given $\bar{y}$, and a much more efficient set of upper bounds can be included implicitly in the algorithm. That is, Bound 10 of section II.6 can be used instead of Bounds 6, 7, 8, and 9. The algorithm changes of section II.6 remain valid with the exception that the new lower bounds in Step 4b' are found from (2.12).

Monotone non-increasing demand

For this case it must be assumed that

$$d_1 - v_o \geq d_2$$

in order that the monotonicity be preserved.

Lemma 2.14 If demands are monotone non-increasing, assumptions (a) and (b) hold and further

$$\text{(c)} \qquad f_j + \sum_{i=j}^{k-1} m_i < c_k \qquad \forall\, k > j,\ j = 1,\ldots,N-1$$

then $w_1 = d_1 - w_0 - v_0$ and $w_i = 0$, $i = 2,\ldots,N$.

Proof: The statement for w_1 results from the fact that no backordering is allowed and by cost assumptions (a) and (b) and monotone demands, it can never pay to build extra capacity to produce early. Assumption (c) eliminates the possibility of scrapping capacity in any period to the extent that capacity must be rebuilt later,which, by monotonicity of demand, produces the result that $w_i = 0$, $i = 2,\ldots,N$.

QED

Lemma 2.15 If demands are monotone non-increasing and $\bar{f} \geq 0$ then $y_N = 0$.

Proof: The proof follows trivially from the fact that demand is monotone non-increasing so, with $\bar{f} \geq 0$, any idle capacity in a period can be salvaged for further revenues.

QED

As was pointed out in section II.6, when

$y_N = 0$, all capacity is used. The transportation subproblem of P5 thus can be solved by inspection for a given $\bar{y}$, speeding up the CIDPI algorithm greatly. The remaining remarks on algorithm changes for this case are the same as for the preceding case of monotone nondecreasing demands.

Chapter III

In this chapter, a special case of the CIDPI model will be presented which leads to a formulation of the entire model as a network. Other extensions of the basic model are then presented to allow piecewise-linear, convex production costs and discontinuities in capacity acquisition.

III.1 Network formulation of the CIDPI model

The formulations of Chapter II required a problem decomposition approach to provide suitable algorithms which took advantage of the problem structure. There is a special case of the CIDPI model, however, where the formulation can be simplified to that of a network and solved with a transshipment algorithm. This occurs when the cost of capacity increase is equal to the salvage value in any period.

Such a model might be useful when there is a large second-hand market in the units of capacity. A firm might then acquire or sell units with equal ease in any period at the going market price. Maintenance costs then reflect the investment needed to keep each unit at its rated capacity for possible future sale.

The derivation of this case comes from reconsideration of the CPI model P1. If $\bar{w} \geq 0$ is dropped, allowing $\bar{w}$ to be unrestricted, the problem would still be bounded if capacity costs c_i, $i = 1,\ldots,N$ were monotone non-increasing, since from constraint (P1.1) and $\bar{x} \geq 0$ it must still be true that

$$\sum_{j=0}^{i} w_j \geq 0 \qquad i = 1,\ldots,N$$

This implies that each w_i is bounded below by the total capacity built up to period i, i.e.

$$w_i \geq -\sum_{j=0}^{i-1} w_j \qquad i = 1,\ldots,N$$

since if the costs are monotone, building and subsequent scrapping of capacity can never result in a profit. That is, with $c_i \geq 0$ and monotone $i = 1,\ldots,N$, the right-hand side will always be as small in absolute value as possible. In fact, as a limit, it could never pay to build more capacity in any period than what is needed to produce all remaining demand in that period, since the excess capacity would be idle and can never reduce the total cost. Thus, with the assumption, which will hold for the remainder of this section, that

$$d_1 - v_o \geq 0$$

a set of lower bounds on $\bar{w}$ can be written as

$$w_1 \geq -w_0$$

$$w_2 \geq v_0 - \sum_{i=1}^{N} d_i$$

$$w_j \geq - \sum_{i=j-1}^{N} d_i \qquad j = 3,\ldots,N$$

Now with $\bar{w}$ unrestricted in P1, if Standard Transformation 1 is performed, then P3 no longer contains constraint set (P3.3). The integrality of $\bar{y}$ still holds, however. To transform the remainder of P3 into a network formulation, slack variables $\bar{s} \geq 0$ must be added to constraint set (P3.1). Then the sum of set (P3.1) is subtracted from the sum of set (P3.2) to give a redundant constraint. Finally, multiplying the constraints of set (P3.2) by -1, the transformed problem can be written as

$$\text{P6:} \qquad \min Z_6 = \sum_{i=1}^{N} [p_i x_i + h_i v_i + c_i' y_i] + K$$

s.t.

$$\text{(P6.1)} \qquad x_1 - y_1 + s_1 = d_1 - v_0$$

$$x_i + y_{i-1} - y_i + s_i = d_i \qquad i = 2,\ldots,N$$

$$-x_1 + v_1 = -d_1 + v_o$$

(P6.2) $$-x_i - v_{i-1} + v_i = -d_i \qquad i = 2,\ldots,N-1$$

$$-x_n - v_{N-1} = -d_N$$

(P6.3) $$y_N - \sum_{i=1}^{N} s_i = 0$$

$$\bar{x}, \bar{y}, \bar{v}, \bar{s} \geq 0 \text{ and } \quad \text{INTEGER}$$

where the constraint (P6.3) is redundant, the c_i' and K are as given in P3, and v_N has been incorporated explicitly in d_N.

The coefficient matrix associated with P6 satisfies the sufficiency conditions for total unimodularity as described by Hu (Hu 69). This guarantees that, since the right-hand sides of the constraints are integer, the linear program associated with P6 without the redundant constraint (P6.3) will terminate integer. P6 posses more useful properties however, since the problem as written can be represented by the network displayed in Figure III-1. Thus the problem can be solved using a transshipment algorithm to obtain the minimum cost flow.

The only evident problems with the network are the cycles involving $\bar{y}$ and $\bar{s}$ which, if negative in cost, cause unboundedness. However, since flow s_i is costless, the monotonicity assumption on the c_i' is

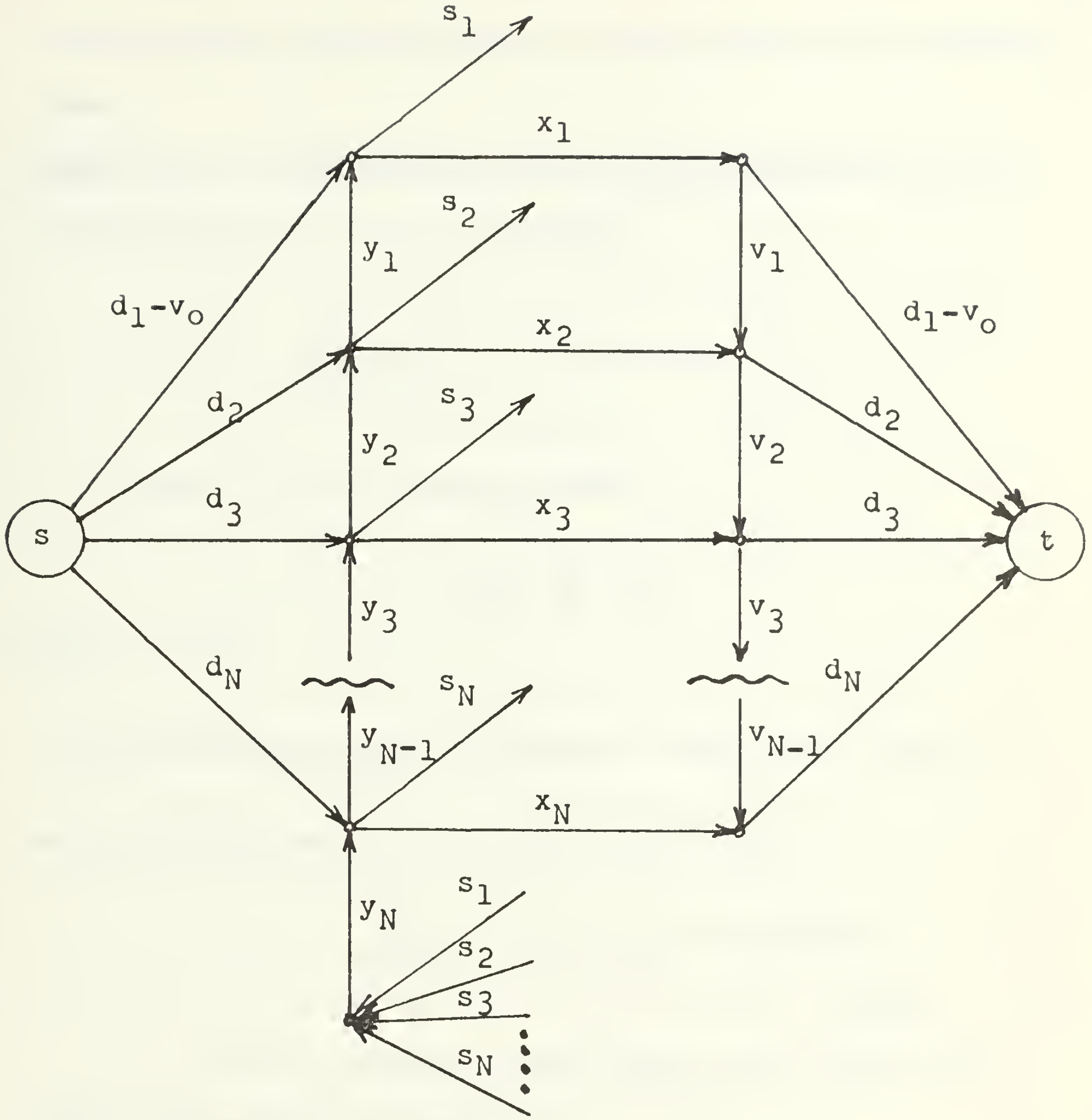
s_1
x_1
s_2
y_1
v_1
d_1-v_o
x_2
d_1-v_o
d_2
s_3
d_2
y_2
v_2
d_3
x_3
d_3
s
t
y_3
v_3
d_N
s_N
d_N
y_{N-1}
v_{N-1}
x_N
y_N
s_1
s_2
s_3
s_N

Figure III-1[1]

Network for model P6

---

[1] The figures on the source and sink arcs represent the arc capacities. For the remaining arcs the figures represent the flow variables associated with the arcs.

sufficient to prevent this as seen from the following lemma.

Lemma 3.1 If capacity costs c'_i are monotone non-increasing for $i = 1,\ldots,N$, then

$$\sum_{i=j}^{N} c'_i \geq 0 \qquad j = 1,\ldots,N$$

Proof: It is obvious that

$$c_N + m_N \geq 0$$

For $j = N-1$

$$(c_{N-1}-2c_N+m_{N-1}-m_N) + (c_N+m_N) = (c_{N-1}-c_N) + m_{N-1} \geq 0$$

and by induction for $j = k$, $1 \leq k < N-1$

$$\sum_{i=k}^{N} c'_i = (c_k-c_{k+1}) + m_k \geq 0$$

QED

It is obvious from Figure III-1 that the only cycles are of the form

$$y_N,\ y_{N-1},\ \ldots,\ y_j,\ s_j \qquad j = 1,\ldots,N$$

which have non-negative costs by Lemma 3.1. This confirms the fact that, since P6 is obtained by a linear transformation on the bounded problem P1, it too must be bounded. Emphasis must also be given to the fact that the capacity change costs c'_i, incorporated in P6, are completely non-stationary and thus admit to dis-

counting or any other aberrant structure.

Networks and the general CPI and CIDPI models

The existence of unimodularity in the special case of P6 permitted use of a network algorithm to obtain an optimal integer solution. The general models, P1 and P2, could also be formulated as networks but equivalence of the networks to the original models requires sets of constraints external to the networks. Moreover, these constraints violate the unimodularity conditions, as shown by the following examples, and thus exclude the use of any existing network algorithm to gain an integer solution. These examples are both linear programs having non-integer solutions with non-integer objective function values (eliminating alternative optima).

Example 1: Example for the CPI model

For $N=6$, $\bar{c} = (20,20,19,19,18,18)$, $\bar{p} = (5,5,4,4,3,3)$, $\bar{h} = (1,1,1,1,1,1)$, $\bar{m} = (3,3,3,3,2,2)$, $\bar{d} = (3,7,9,13,18,21)$, $w_o = 1$, $v_o = 1$

LP optimum

$x_1 = 2$, $x_2 = 7$, $x_3 = x_4 = x_5 = x_6 = 15.25$

$v_3 = 6.25$, $v_4 = 8.5$, $v_5 = 5.75$

$w_1 = 1$, $w_2 = 5$, $w_3 = 8.25$

Objective Function = 735.25

Example 2: Example for the CIDPI model

The problem is as in Example 1 with the exception of $\bar{f} = (8,8,7,7,6,6)$, $\bar{d} = (3,5,8,14,9,7)$

LP optimum

$x_1 = x_2 = x_3 = x_4 = x_5 = 7.6$, $x_6 = 7.0$

$v_1 = 5.6$, $v_2 = 8.2$, $v_3 = 7.8$, $v_4 = 1.4$, $v_5 = 0$

$w_1 = 6.6$ $r_6 = .6$

Objective Function = 457.2

III.2 Block Capacity

In this section, CPI and CIDPI models will be extended to handle the case of block capacity. This terminology embraces any situation where capacity can only be acquired in units which are each capable of producing large numbers of units of demand in each period, as opposed to the implicit assumption in effect so far, that each unit of capacity produced a single demand unit per period. Block capacity, then, corresponds to the economists' problem of "lumpiness" in acquisition which typically causes serious difficulty in a marginal analysis approach.

In the remainder of this section it is assumed

that in any period

1 Capacity Unit = k Demand Units

where k is a positive integer and is stationary, and that initial capacity, w_o, is expressed in units of capacity.[2]

CPI-Block capacity model

Working first on the CPI model, in terms of block capacity, (2.1) can be rewritten as

$$kw_1 - v_1 \geq d_1 - (kw_o+v_o)$$

$$\vdots$$

$$k(Nw_1+\ldots+w_N) - v_N \geq d_1 + \ldots + d_N - (Nkw_o+v_o)$$

Following the same development as in Chapter II, and using Standard Transformation 1 to solve for $\bar{w}$ gives

$$w_1 = \frac{1}{k}[y_1 + d_1 - (kw_o+v_o)]$$

$$w_2 = \frac{1}{k}[y_2 - 2y_1 + d_2 - d_1 + v_o]$$

$$w_N = \frac{1}{k}[y_N - 2y_{N-1} + y_{N-2} + d_N - d_{N-1}]$$

---

[2] Depending on the source of the demand, management has the option of choosing the most appropriate unit for demand, e.g. box, pallet-load, car-load, etc. The

and the expressions for w_i can now be substituted back into P1 to get the transformed problem. In this case, substitution in the $\bar{W} \geq 0$ constraints gives somewhat expanded results. The constraint set corresponding to (P3.3) now looks like

$$\frac{1}{k}\,[y_1 + d_1 - (kw_o + v_o)] \geq 0 \quad \text{and INTEGER}$$

$$\vdots$$

$$\frac{1}{k}\,[y_N - 2y_{N-1} + d_N - d_{N-1}] \geq 0 \quad \text{and INTEGER}$$

Taking the first constraint and ignoring the integer requirement gives

$$y_1 \geq kw_o + v_o - d_1$$

while the integer requirement gives

$$\frac{1}{k}\,[y_1 + d_1 - (kw_o + v_o)] = n_1$$

where n_1 is a non-negative integer. But this is the same as

$$y_1 = n_1 k - d_1 + kw_o + v_o$$

---

value of k can then be set appropriately.

A similar development for period j gives

$$y_j \geq 2y_{j-1} - y_{j-2} + d_{j-1} - d_j$$

and

$$y_j = n_j k + 2y_{j-1} - y_{j-2} + d_{j-1} - d_j$$

for $n_j \geq 0$ and integer. So, in general, substitution in $\bar{w} \geq 0$ gives two sets of constraints, the first similar to set (P3.3)

$$\begin{aligned} & y_1 \geq kw_o + v_o - d_1 \\ (3.1) \quad & y_2 \geq 2y_1 + d_1 - d_2 - v_o \\ & y_i \geq 2y_{i-1} - y_{i-2} + d_{i-1} - d_i \qquad i = 3,\ldots,N \\ & \bar{y} \geq 0 \end{aligned}$$

and the second, a set of integrality constraints on $\bar{y}$

$$\begin{aligned} & y_1 = n_1 k - d_1 + kw_o + v_o \\ (3.2) \quad & y_2 = n_2 k + 2y_1 + d_1 - d_2 - v_o \qquad n_i \geq 0 \text{ and INTEGER} \\ & \qquad \vdots \qquad\qquad\qquad\qquad\qquad i = 1,\ldots,N \\ & y_N = n_N k + 2y_{N-1} - y_{N-2} + d_{N-1} - d_N \end{aligned}$$

The effect of k, then is to superimpose a lattice of

interval k on the space of $\bar{y}$. Up until now, $k = 1$ and so the enumeration on $\bar{y}$ had to search every feasible integer point in $\bar{y}$ space. As k becomes larger, however, the number of enumerable points in $\bar{y}$ space becomes smaller for a given $\bar{d}$ and the problem can be solved more quickly.

Doing the remaining substitution for $\bar{w}$ in P1 gives the following form of the CPI-Block capacity model.

$$\text{P7:} \quad \min Z_7 = \sum_{i=1}^{N} [p_i x_i + h_i v_i] + \frac{1}{k}\Big[\sum_{i=1}^{N} c'_i y_i + K\Big] - \frac{(k-1)}{k} c_1 w_o$$

s.t.

$$(\text{P3.1}),\ (\text{P3.2}),\ (3.1),\ \text{and}\ (3.2)$$

$$\bar{x},\ \bar{v} \geq 0 \quad \text{and INTEGER}$$

$$\bar{y} \geq 0$$

where c'_i and K are as in P3. The basic algorithm is valid for P7 with a few changes. In step 4b let $y_j = y_j + k$. The global upper bounds are changed slightly by the parameter k, since the maximum value for w_1 is now

$$w_1 = \frac{1}{k}\Big(\sum_{i=1}^{N} d_i - v_o\Big) - w_o$$

So using (3.2), (2.5) of Chapter II becomes

$$(3.3) \qquad y_j \leq jk \cdot \max w_1 - \sum_{i=1}^{j} d_i + v_o \qquad j = 1, \ldots, N$$

In order to satisfy (3.1), (3.2) and $\bar{y} \geq 0$, the lower bounds on $\bar{y}$ are first obtained from (3.1). If $\bar{y} \geq 0$, then (3.2) is also satisfied. If $y_i < 0$, $1 \leq i \leq N$, then by adding $m_i k$ to y_i for successive values of $n_i = 1, 2, 3, \ldots,$ until $y_i \geq 0$, a feasible lower bound can be obtained which satisfies (3.2).

CIDPI-Block capacity model

Now for the CIDPI model, (2.11) can be rewritten in terms of block capacity as

$$(w_1 - r_1) = \frac{1}{k}[d_1 + y_1 - (kw_o + v_o)]$$

$$(w_2 - r_2) = \frac{1}{k}[d_2 - d_1 - 2y_1 + y_2 + v_o]$$

$$(w_i - r_i) = \frac{1}{k}[d_i - d_{i-1} + y_{i-2} - 2y_{i-1} + y_i] \qquad i = 3, \ldots, N$$

Substituting for $\bar{r} \geq 0$ in P2 gives

$$w_1 - \frac{1}{k}[d_1 + y_1 - (kw_o + v_o)] \geq 0 \qquad \text{and INTEGER}$$

$$\vdots$$

$$w_N - \frac{1}{k}[d_N - d_{N-1} + y_N - 2y_{N-1} + y_{N-2}] \geq 0 \qquad \text{and INTEGER}$$

Following a development similar to the first part of this section, the jth constraint is equivalent to the two constraints

$$w_j \geq \frac{1}{k}(d_j - d_{j-1} + y_j - 2y_{j-1} + y_{j-2})$$

and

$$y_j = n_j k + d_{j-1} - d_j + 2y_{j-1} - y_{j-2}$$

where n_j is any integer, but now $n_j < 0$ is allowable since the right-hand side of the first constraint is no longer required to be a non-negative.

In general, the two constraint sets which result are

$$\begin{aligned} & w_1 \geq \frac{1}{k}[d_1 + y_1 - (w_o + v_o)] \\ (3.4) \qquad & w_2 \geq \frac{1}{k}[d_2 - d_1 - 2y_1 + y_2 + v_o] \\ & w_i \geq \frac{1}{k}[d_i - d_{i-1} + y_i - 2y_{i-1} + y_{i-2}] \qquad i = 3,\ldots,N \end{aligned}$$

and

$$\begin{aligned} & y_1 = n_1 k - d_1 + (kw_o + v_o) \\ (3.5) \qquad & \\ & y_2 = n_2 k + 2y_1 + d_1 - d_2 - v_o \end{aligned}$$

$$y_3 = n_3k + 2y_2 - y_1 + d_2 - d_3 \qquad n_i \quad \text{INTEGER}$$

$$\vdots \qquad\qquad i = 1,2,\ldots,N$$

$$y_N = n_Nk + 2y_{N-1} - y_{N-2} + d_{N-1} - d_N$$

Substitution for $\bar{r}$ in the remainder of P2 just repeats the first three constraint sets of P5 and gives the model

$$\text{P8:} \qquad \min Z_8 = \sum_{i=1}^{N} [p_i x_i + h_i v_i + f''_i y_i + (c_i - f_i) w_i] + M'$$

s.t.

(P5.1), (P5.2), (3.4), and (3.5)

$$\bar{x}, \bar{v}, \bar{w} \geq 0 \quad \text{and INTEGER}$$

$$\bar{y} \geq 0$$

where

$$f''_i = \frac{1}{k} f'_i$$

$$M' = \frac{1}{k} M - \frac{(k-1)}{k} f_1 w_o$$

for f'_i and M from P5.

Modifications to the basic algorithm for P8 are the same as for P7 except that, if the lower bound for any y_i is positive, then $(n_i k)$ should be subtracted from y_i for successive values $n_i = 1,2,3,\ldots,$ until the

smallest $y_i \geq 0$ is obtained. This corresponds to the unrestricted n in constraint set (3.5) which is caused by the possibility of having $r_i > 0$ in any period.

It should be noted here that the special cases of Chapter II are valid here only for k = 1. Another point worth noting is the increase in danger of rounding off from the related linear programming solution as k increases. Not only are the unit capacity costs larger for large k, but also shifting a unit from one period to another has a proportionally greater effect on the optimal production-inventory schedule.

Finally, the increase in the lattice interval with k leads to a useful scheme when deciding on a capacity increase (and decrease) strategy. If several sizes of capacity are available, then the algorithm can be run for each value of k with the running time probably decreasing for each larger value of k. This allows a third dimension of optimization for management with the cost of gaining the added information governed by the algorithm running time for the smallest value of k. Note, however, that the form of Standard Transformation 1 does not permit k to be non-stationary within a model.

III.3 Convex Production Cost

Up until now, it has been assumed that all

production was being carried out by one shift working regular hours with unit production costs $\bar{p}$. In this section the CPI and CIDPI models with block capacity will be extended to allow multi-shift and overtime production. Although the work done here is for one additional shift, further extensions are trivial, as will be pointed out later.

The crucial assumption of this section is that the unit cost of production increases for each shift and for overtime. In the manufacturing context, this assumption is not restrictive and actually models reality, since it is commonly accepted that there exist "shift differentials" in manufacturing costs for a second and third shift. These can be ascribed, in part, to the increased time spent on maintenance and to greater human error in succeeding shifts. As for overtime production, the wage differential alone is sufficient to justify the assumption.

Let

x_i' - second shift production in period i

p_i' - second shift unit production cost in period i

Production capacity has been defined in terms of regular time production. It is assumed that, in terms of the second shift, an equal amount of capacity is

available. Thus

$$x'_i \leq \sum_{j=1}^{i} w_j + w_o \qquad i = 1,\ldots,N$$

Assume that

$$p_i \leq p'_i \qquad i = 1,\ldots,N$$

Then the production cost function for period i

$$P_i(x_i,x'_i) = p_i x_i + p'_i x'_i$$

is piecewise-linear convex. Similarly, with a suitable rearrangement of terms, total production cost

$$P = \sum_{i=1}^{N} P_i(x_i,x'_i)$$

is piecewise-linear convex.

CPI-Block capacity- Convex production cost model

First the extension of the CPI model with block capacity will be presented. The term $\sum_{i=1}^{N} p'_i x'_i$ must be added to the objective function of P_1, while the following constraints are added to the constraint set:

(P1.1a)
$$x'_i - k\sum_{j=1}^{i} w_j \leq kw_o \qquad i = 1,\dots,N$$
$$\bar{x}' \geq 0 \quad \text{and INTEGER}$$

and the appropriate block capacity coefficients are inserted in the remainder of the model. Then (P1.1), (P1.1a), and (P1.2) can be combined to give a set of necessary and sufficient conditions for feasibility which take into account two-shift production.

$$\begin{aligned}
2kw_1 - v_1 &\geq d_1 - v_o - 2kw_o \\
2k(2w_1+w_2) - v_2 &\geq d_1 + d_2 - v_o - 4kw_o \\
&\vdots \\
2k(Nw_1+\dots+w_N) - v_N &\geq \sum_{i=1}^{N} d_i - v_o - 2Nkw_o
\end{aligned}$$

Again by Standard Transformation 1 a solution for $\bar{w}$ can be obtained. Then substituting into $\bar{w} \geq 0$ of P1 gives the following two sets of constraints

(3.6)
$$\begin{aligned}
y_1 &\geq v_o + 2kw_o - d_1 \\
y_2 &\geq 2y_1 + d_1 - d_2 - v_o \\
y_i &\geq 2y_{i-1} - y_{i-2} + d_{i-1} - d_i \qquad i = 3,\dots,N
\end{aligned}$$

and

(3.7)

$$y_1 = 2n_1k - d_1 + 2kw_o + v_o$$

$$y_2 = 2n_2k + 2y_1 + d_1 - d_2 - v_o$$

$$y_3 = 2n_3k + 2y_2 - y_1 + d_2 - d_3 \qquad n_i \geq 0 \text{ and INTEGER}$$

$$\vdots \qquad i = 1,2,\ldots,N$$

$$y_N = 2n_Nk + 2y_{N-1} - y_{N-2} + d_{N-1} - d_N$$

From (3.7) it is clear that the lattice interval on the $\bar{y}$ space has been increased to 2k with the introduction of a second shift.

Completing the substitution and transformation from P1 gives

P9: $$\min Z_9 = \sum_{i=1}^{N} [p_ix_i + p'_ix'_i + h_iv_i + c''_iy_i] + k'$$

s.t.

(P9.1)

$$x_1 \leq \frac{1}{2}(d_1 + y_1 - v_o)$$

$$x_i \leq \frac{1}{2}(d_i + y_i - y_{i-1}) \qquad i = 2,\ldots,N$$

(P9.2)

$$x'_1 \leq \frac{1}{2}(d_1 + y_1 - v_o)$$

$$x'_i \leq \frac{1}{2}(d_i + y_i - y_{i-1}) \qquad i = 2,\ldots,N$$

(P9.3) $$x_i + x'_i + v_{i-1} - v_i = d_i \qquad i = 1,\ldots,N$$

(3.6)

(3.7)

$$\bar{x},\ \bar{x}',\ \bar{v} \geq 0 \quad \text{and INTEGER}$$

$$\bar{y} \geq 0$$

where

$$c''_i = \frac{1}{2k} c'_i \qquad i = 1,\dots,N$$

and

$$K' = \frac{1}{2k} K - \frac{(2k-1)}{2k} c_1 w_o$$

for c'_i and K from P3.

The addition of the extra term to the objective function and the extra set of constraints modifies the transportation problem structure of the subproblem. In effect, N additional sources have been added and the number of arcs has been doubled. This development might make it beneficial to reformulate the sub-problem as a transshipment problem which can be solved with an appropriate minimum cost flow algorithm (Ford-Fulkerson 62). An investigation would have to be made of the tradeoff between the faster algorithm but higher network growth rate of the transportation problem against the slower algorithm but lower network growth rate of the transshipment problem. Other algorithm changes and remarks on the bounds for $\bar{y}$ are

identical to those for the CPI model of the previous section.

CIDPI-Block capacity- Convex production cost model

The convex production cost case of the CIDPI model with block capacity can be obtained easily by a similar development. In P2, the term $\sum_{i=1}^{N} p_i' x_i'$ is added to the objective function, the constraints

$$\text{(P2.1a)} \qquad x_i' - k \sum_{j=1}^{i} (w_j - r_j) \leq kw_o \qquad i = 1, \ldots, N$$

$$\bar{x}' \geq 0 \quad \text{and INTEGER}$$

are added to the constraint set and the appropriate block capacity coefficients are inserted in the remainder of the model. A set of necessary and sufficient conditions for feasibility is then obtained from (P2.1), (P2.1a), and (P2.2) as

$$2k(w_1 - r_1) - v_i \geq d_1 - v_o - 2kw_o$$

$$2k[2(w_1 - r_1) + (w_2 - r_2)] - v_2 \geq d_1 + d_2 - v_o - 4kw_o$$

$$\vdots$$

$$2k[N(w_1 - r_1) + \ldots + (w_N - r_N)] - v_N \geq \sum_{i=1}^{N} d_i - v_o - 2Nkw_o$$

Using Standard Transformation 1 to solve for $\bar{r}$ and

substituting for $\bar{r} \geq 0$ in P2, the following two sets of constraints, analogous to (3.4) and (3.5), are obtained.

$$
\begin{aligned}
w_1 &\geq \frac{1}{2}k[d_1+y_1-v_o-2kw_o] \\
w_2 &\geq \frac{1}{2}k[d_2-d_1-2y_1+v_o] \\
w_i &\geq \frac{1}{2}k[d_i-d_{i-1}+y_i-2y_{i-1}+y_{i-2}] \qquad i = 3,\ldots,N
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
y_1 &= 2n_1k - d_1 + v_o + 2kw_o \\
y_2 &= 2n_2k + 2y_1 + d_1 - d_2 - v_o \\
y_3 &= 2n_3k + 2y_2 - y_1 + d_2 - d_3 \qquad n_i \text{ INTEGER} \\
&\;\;\vdots \qquad\qquad\qquad\qquad\qquad\qquad i = 1,2,\ldots,N \\
y_N &= 2n_Nk + 2y_{N-1} - y_{N-2} + d_{N-1} - d_N
\end{aligned}
\tag{3.9}
$$

Again note that the lattice interval on the $\bar{y}$ space has been increased to 2k.

Completing the substitution for $\bar{r}$ in P2, the model is transformed to the following:

$$
\text{P10: } \min Z_{10} = \sum_{i=1}^{N} [p_i x_i + p'_i x'_i + h_i v_i + f'''_i y_i + (c_i - f_i) w_i] + M''
$$

s.t.

(P10.1)
$$x_1 \le \frac{1}{2}(d_1 + y_1 - v_o)$$
$$x_i \le \frac{1}{2}(d_i + y_i - y_{i-1}) \qquad i = 2,\dots,N$$

(P10.2)
$$x'_1 \le \frac{1}{2}(d_1 + y_1 - v_o)$$
$$x'_i \le \frac{1}{2}(d_i + y_i - y_{i-1}) \qquad i = 2,\dots,N$$

(P10.3)
$$x_i + x'_i + v_{i-1} - v_i = d_i \qquad i = 1,\dots,N$$

(P10.4)
$$d_1 + y_1 - v_o \ge 0$$
$$d_i + y_i - y_{i-1} \ge 0 \qquad i = 2,\dots,N$$

(3.8)

(3.9)

$$\bar{x},\ \bar{x}',\ \bar{v},\ \bar{w} \ge 0 \quad \text{and INTEGER}$$

$$\bar{y} \ge 0$$

where

$$f'''_i = \frac{1}{2k}f'_i \qquad i = 1,\dots,N$$

and

$$M'' = \frac{1}{2k} M - \frac{(2k-1)}{k} f_1 w_o$$

where f_1' and M are from P5.

The remarks of the previous section relevant to the changes in the algorithm and in the bounds apply here also. As noted there, the subproblem can be formulated as either a transportation problem or a transshipment problem.

As has been mentioned, the techniques of this section easily extend the models to encompass a third shift and overtime work. The capacity available for overtime can be assumed to be equal to that for one regular time shift. This presents a problem since overtime usually includes both time-and-one-half and double-time wage periods. Some subjective unit production cost would have to be chosen to represent the entire period rather than differentiating between the two cost levels.

Finally, as was already remarked, an addition of another shift increased the lattice interval on the $\bar{y}$ space. If the extension is continued, then the first term on the right-hand sides of (3.6) and (3.8) becomes 3nk with a third shift, and 4nk with overtime, increasing the interval by nk each time.

Chapter IV

The basic idea of the enumeration algorithms presented is similar to work done by Murty (Murty 68) and Gray (Gray 71). A discussion of these related approaches opens this chapter. Also included are a short discussion of the restart capability of the algorithms and a presentation of computational results. Finally, some conclusions regarding the preceding work and suggestions for possible extensions are given.

IV.1 The Decomposition Approach to Integer Programming

Since the algorithms developed in Chapter II use the idea of decomposition, it would be useful at this point to put them in perspective relative to some similar approaches in integer programming. This is not done to claim extensions to existing theory, but rather to show contrasts and similarities occuring when a common approach is used on different problem structures.

One approach which uses the properties of duality theory to develop a gradient algorithm for mixed integer programming was given by Benders (Hu 69). A drawback to the algorithm is that efficient use of the gradient information requires the solution of a series

of integer programs where the number of constraints is increasing by one at each iteration. The algorithms of Chapter II only make partial use of the available gradient information in the convexity properties which are developed. Possible further use of this information was not investigated in the course of this work.

Other approaches more closely related in principle and in spirit to the algorithms of this thesis are the work of Murty (Murty 68) and Gray (Gray 71). Both works deal with an exact solution to the fixed charge problem and, as pointed out by Gray, the two have mutually enhancing properties and could probably be combined to give a powerful algorithm.

Murty develops a decomposition approach which ranks the extreme points of the convex hull according to the value of the variable cost. An upper bound on variable cost is given by the use of a minimum value for the fixed cost in a manner similar to that which gives an upper bound on capacity cost in the algorithms of this thesis. The variable cost upper bound can then be used to terminate the enumeration of extreme points. The algorithm's efficiency is greatest when the fixed costs are small both in absolute and relative magnitude with respect to the variable costs (Gray 71).

Up until now, no mention has been made of the

fixed charge aspects of P1 and P2. It is not hard to make such an interpretation of the capacity costs however, since once incurred, they are spread over the units produced in all future periods. The capacity cost of this thesis can thus be thought of in terms of a fixed charge although the problem formulation does not make this explicit.

In light of the above, the approach of this thesis is much closer to that of Gray than of Murty, since Gray's decomposition approach works mainly on the fixed cost section of the problem and deals with the variable cost as a subproblem. His fixed cost section is concerned with the optimal profile of open and closed routes of a transportation problem where each open route incurs a fixed charge. The scheme used is enumeration in a number system of base 2^n(where n is the number of sinks) which takes advantage of both the transportation problem structure and the 0-1 property of the integer variables. In Gray's approach, an upper bound is developed on the fixed cost which is analogous to $\hat{\alpha}$ of Chapter II.

The major difference between Gray's (and Murty's) model and those of this thesis is the 0-1 nature of the integer variables. Transformation of P1 and P2 to 0-1 variables could be done, but at the

expense of a huge increase in the number of variables.

Gray also encounters another difficulty not present in this thesis. He has no means of ensuring that each node of the enumeration is feasible to the to the main problem, thus expending resources in enumerating, and testing for, infeasible nodes. As has been mentioned previously, enumeration on $\bar{y}$ assures feasibility at every node in the algorithms of this thesis.

IV.2 Restart Capability of the Algorithm

The algorithms developed have a feature that enhances their flexibility and usefulness, that is the capability of starting them at any point in the enumeration. This feature is inherent in the method of enumerating on $\bar{y}$; since given any $\bar{y}'$ and the value of the algorithm pointer (PNT) plus some value for the current bound on capacity cost $(\hat{\alpha})$ and the current optimal cost, the algorithm can be started from Step 5 and run normally. Even if $\hat{\alpha}$ and current optimal cost are set to infinity, the next execution of Steps 6 through 9 will establish a new set of bounds, though if the optimal solution precedes the point at which the algorithm is restarted, the new bounds may not be as tight as those which would have resulted from the

optimal solution. If the optimal solution is given by some vector after $\bar{y}$ it will be found whether or not initial bounds are provided.

This restart capability, when combined with the linear programming bound, provides a very useful tool to management, especially when a large problem which could be expected to have a long running time is encountered. Instead of starting the algorithm from the lower bounds of $\bar{y}$, a $\bar{y}'$ corresponding to a $\bar{w}$ slightly lower than any rounded-off solution to the linear program can be chosen as a starting point. The algorithm can then be run for a sufficient period of time to inspect all $\bar{w}$ which could have been rounded-off from the linear programming optimum. An optimal solution is assured here if it equals $\lceil Z^*_{LP} \rceil$. If none such is found, then a management decision can be made based on the difference between the current optimum and Z^*_{LP} as to whether the full algorithm should be run in an attempt to achieve further improvement.

It might also be advantageous, e.g. because of limited computer time, to run the algorithm for relatively short periods of time. If no terminal condition has been reached then, again, a management decision can be made on whether to attempt to improve the current optimal solution with respect to Z^*_{LP}.

If so, the algorithm can be started right where it last terminated with no loss of efficiency.

IV.3 Employment Smoothing

The relevance of the literature on employment smoothing was mentioned in Chapter I. Now that the development has been completed in the context of capacity expansion, a reinterpretation of the results can be made to handle the employment smoothing problem (O'Malley et.al. 66, Silver 67, Lippman et.al 67a and 67b, Lippman-Yuan 69). Reference, in this section, will be to the most general models developed, i.e. those of section III.3, from which all special cases can follow trivially.

The time period for the employment smoothing problem is much shorter than that for capacity expansion by the simple fact that hiring, firing, and usually even training, labor does not take as long as changing fixed capacity. It is assumed, therefore, that fixed capital capacity is never a constraint on the size of the work force.

The work force units can be either in terms of individuals, or groups such as gangs, crews, etc. It is assumed that there is a capacity associated with each labor unit which can be expressed in terms of the

consumptive factor bundles ∂_1. This capacity could be gotten from the associated fixed capital optimal operating level capacity or by a separate marginal productivity level argument for labor as was done for capital in Chapter I. It is again assumed that the production function exhibits constant returns to scale in labor and the consumptive factor. The accompanying assumption is that the existing labor force can be used at any level up to its capacity with no lost opportunity cost of idle labor other than wages.

A change in the factors of capacity will have to be made. Before, the capacity definition was expressed in terms of one shift doing regular time work. Now capacity is defined in terms of all possible regular time output which can be achieved by the existing work force. Thus it is assumed that the shifts are filled sequentially, i.e. before adding second shift labor, a full first shift must be operating.

The CPI model P9 would have limited application in the employment smoothing sense because it ignores capacity reduction. Its simplifying features relative to the CIDPI model however, make its use attractive when the circumstances permit. Two instances when such use would be feasible are those of a firm phasing in a new production line where the phase-in

period is relatively long, say a year; and that of a firm, with a reduced work force on an existing line, facing a projected growth in demand.

Reinterpretation of P9 to employment smoothing leaves the inventory variables $\bar{v}$ and their coefficients $\bar{h}$ unchanged. The production variables $\bar{x}$ now represent all regular time production, and no differentiation in production costs $\bar{p}$ among shifts is allowed. Overtime production is $\bar{x}'$ and again undifferentiated overtime production costs $\bar{p}'$ must be assigned. The capacity expansion variables $\bar{w}$ now become work force hiring variables and the associated costs $\bar{c}$ can include all the normal costs associated with hiring personnel, i.e. advertising, personnel staff costs, training, etc. Wages now take the place of maintenance costs for $\bar{m}$.

As the reader has probably noticed, an interpretational difficulty exists concerning the capacity ascribed to overtime production in P9. The assumption for that model was that overtime capacity was approximately equal to the capacity of a regular time shift. What is necessary now, if Standard Transformation 1 is to be used, is the seemingly ludicrous assumption that overtime capacity is equal to all regular time capacity. With a time span of one to one and one-half years, however, discounting will not be significant so that over-

time production costs will be much higher than any regular time costs over the period. Furthermore, hiring costs are now in the same order of magnitude as production costs so that, within a reasonable range, hiring additional regular time labor should be cheaper than working any significant amount of overtime. A reasonable approach to the problem would be to first solve it for regular time only, and then try a solution permitting overtime. If the overtime assumption causes difficulty, then either a subjective modification of costs can be made and the problem re-solved or the approach has to be abandoned. For the majority of cases it would seem that the approach is useful.

For the more realistic CIDPI model P10, the only additional interpretation necessary is that the $\bar{r}$ now represent the firing variables and the $\bar{f}$ the firing costs. The same difficulties exist for overtime capacity and the previous remarks apply equivalently here.

An argument can be made for the equivalence of hiring and firing costs in any period. Certainly what is included and excluded from each cost is subjective. Assuming that equality exists and, further, that the $\bar{c}$ are monotone non-increasing with time, then the special case of the network formulation is applicable to the employment smoothing problem with the

preceding interpretations on the relevant variables where in addition $w_1 < 0$ corresponds to the firing variable $r_1 > 0$ of P10. Moreover, use of the network excludes the possibility of overtime and requires that $k = 1$. No obvious extension of the network model to remove these restrictions exists.

Finally, the assignment of a value for k in P9 and P10 will undoubtedly be more difficult in terms of labor than it was in terms of fixed capital. On the other hand, the sensitivity of the models to k should not be significant for labor because the hiring and firing costs are much smaller than the corresponding capacity change costs, and thus the impact of fluctuations in the labor force level will not be as great.

This work, then, becomes the first exact integer approach for the employment smoothing problem.[1] The algorithms given for the special demand cases in sections II.6 and II.9 are more general than those of Lippman et.al. (Lippman et.al. 67b), and algorithms to handle general demands and general linear costs have

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[1] As was mentioned in Chapter I, a "round-off" approach from a linear programming model has been presented (O'Malley et. al. 66).

been given in addition. Finally, the similarity of the two areas of capacity expansion and employment smoothing certainly suggests that a combined effort would be useful.

IV.4 Computational Results

The algorithms developed in this thesis were coded in Fortran IV and run on the IBM 7094 in the Johns Hopkins University computing center. The test problems used were set up by the author using non-random data. Twelve problems were generated in all, each one as a ten-period problem. When run, however, each problem was run eight times using the first k periods of data each time for k = 3,...,10. Thus e.g. for problem 10, the three period problem is numbered 10.1, while the ten period problem is numbered 10.8. This numbering system is followed in the tables of data which follow.

The following conditions are satisfied by all of the problems: a) Costs are monotone non-increasing with time. In addition, the costs of Problems 7, 8, 9 and 10 are stationary. b) k = 1 (1 Capacity unit = 1 demand unit). c) Only one shift of regular time work is being used.

A linear program was solved for the CPI and

CIDPI case of each problem (using models P1 and P2). Because of the "nearly" unimodular structure of the general models, a large number of integer extreme points probably exist in each convex hull. In fact, for the 166 linear programs solved for Problems 1 and 3 to 12, a total of 40 terminated integer.

It had been thought that the number of periods would be the most significant factor in problem running time and that the appropriate curves could be presented. Instead, the models turn out to be sensitive to both the cost and demand structures of the individual problems, as well as being dependent on the number of periods. The computational results are, therefore, presented in tabular form in Tables IV-1, IV-2, and IV-3 at the end of this section.

An example of the sensitivity of the algorithms to demand and cost structure is given by Problems 9 and 12 using the CPI algorithm. The data for the problems is as follows:

Problem 9

$\bar{d} = (5, 7, 11, 6, 10, 14, 9, 13, 19, 16)$

$\bar{c} = (30, 30, \ldots, 30)$

$\bar{m} = (3, 3, \ldots, 3)$

$\bar{p} = (5, 5, \ldots, 5)$

$\bar{h} = (1, 1, \ldots, 1)$

Problem 12

$\bar{d}$ = (2, 3, 5, 4, 6, 8, 7, 9, 12, 10)

$\bar{c}$ = (250, 240, 230, 220, 210, 200, 190, 180, 170, 160)

$\bar{m}$ = (3, 3, ..., 3)

$\bar{p}$ = (5, 5, 5, 4, 4, 4, 4, 3, 3, 3)

$\bar{h}$ = (1, 1, ..., 1)

For t = running time

n = number of periods

a curve fitting program was run for a polynomial

$$t = an^b$$

and an exponential

$$t = ae^{bn}$$

fit on the data. For Problem 9, using 7 data points the results were:

Polynomial

$a = .00028$ $b = 5.10$

Coefficient of correlation = .960

Exponential

$a = .00635$ $b = .94$

Coefficient of correlation = .606

while using 7 data points for Problem 12 the results were:

Polynomial

a = .00006 b = 5.63

Coefficient of correlation = .875

Exponential

a = .00573 b = .87

Coefficient of correlation = .973

When the data for 6 common points was averaged the following resulted:

Polynomial

a = .00003 b = 6.21

Coefficient of correlation = .561

Exponential

a = .00339 b = .993

Coefficient of correlation = .770

Similar results were obtained using the data from the other problems for both the CPI and CIDPI cases.

The only conclusion which can be drawn,

then, is that the number of periods involved is a significant factor in problem running time, but there exist other significant factors. Inspection of the tables will verify the claim made at the end of section II.8 that longer running times should result for the CIDPI algorithm.

Two interesting examples remain to be shown, the first demonstrating the acknowledged fallacy of rounding-off the linear programming solution, and the second a demonstration of the power of the restart capability discussed in section IV.2.

Example 1: For the CPI case of Problem 12.2, the variable values resulting from the linear and integer programming optima respectively are (with $w_o = 1$ and $v_o = 1$)

LP $w_1 = 2.25, \quad w_2 = w_3 = w_4 = 0$
$x_1 = x_2 = x_3 = x_4 = 3.25$
$v_1 = 2.25, \quad v_2 = 2.5, \quad v_3 = .75, \quad v_4 = 0$

IP $w_1 = 0, \quad w_2 = 3, \quad w_3 = w_4 = 0$
$x_1 = 1, \quad x_2 = x_3 = x_4 = 4$
$v_1 = 0, \quad v_2 = 1, \quad v_3 = v_4 = 0$

The difference between building all capacity in period 1 and building it all in period 2 could be significant

to management. Furthermore, the production-inventory policies are quite different in each case.

Example 2: The CPI algorithm for Problem 8.8 took 27.067 seconds to terminate where termination resulted from the linear programming bound. The optimal capacity expansion vector for the linear programming solution was

$$\bar{w} = (1,\ 4,\ 3.875,\ 0,\ \ldots,\ 0)$$

The CPI algorithm was started with a $\bar{y}$ corresponding to $w_1 = 1$, $w_2 = 4$, $w_3 = 3$, and the remaining y_4 through y_{10} set at their lower bounds. The time to termination by the linear programming bound from this starting point was 5.9167 seconds.

In terms of core usage in the computer, the algorithms are limited only by the requirements of the transportation problem. The version of the transportation algorithm used requires only one mxn array where for the algorithms of Chapter II, $m = n = N$, while if a transportation algorithm is used for the convex production cost use of section III.3, $m = 2N$, and $n = N$. The enumeration itself is performed using a single vector N+2 words in length. The growth in core requirements is therefore dominated entirely by the growth of the transportation array.

Table IV-1

Computational results for the CPI algorithm with monotone non-decreasing demand

Problem Number	Periods (N)	Optimal Values		Run Time(sec)		Step 8 Termination [2]
		LP	IP	LP	IP	
1.2	4	385.0	388	.73	.383	No
6.2	4	176.0	179	.77	.20	No
7.2	4	211.0	217	.77	.20	No
1.3	5	557.0	558	1.05	1.533	No
6.3	5	250.0	252	1.03	.40	No
7.3	5	309.0	317	1.05	.317	No
1.4	6	735.25	736	1.33	.733	Yes
6.4	6	335.25	336	1.33	.283	Yes
7.4	6	429.50	437	1.47	.717	No
1.5	7	884.60	885	1.83	1.383	Yes
6.5	7	418.40	420	1.82	3.083	No
7.5	7	572.85	578	2.35	2.467	No
1.6	8	1052.0	1055*	2.27	8^+ min	
7.6	8	739.75	741	3.02	11.383	No
6.7	9	583.60	586	3.50	85.233	No
7.7	9	940.75	942	3.75	75.350	No
6.8	10	668.25	670*	4.23	11^+ min	
7.8	10	1173.0	1174	4.67	474.50	No

*This was the current optimal solution when the enumeration was stopped.

[2] Step 8 uses $\lceil Z^*_{LP} \rceil$ as a termination rule for the enumeration.

Table IV-2

Computational results for the CPI algorithm with general demand

Problem Number	Periods (N)	Optimal Values		Run Time(sec)		Step 8 Termination
		LP	IP	LP	IP	
4.1	3	299.33	310	.40	.283	No
9.1	3	373.0	391	.40	.20	No
4.2	4	472.0	478	.73	.883	No
8.2	4	271.0	277	.75	.317	No
9.2	4	425.0	445	.72	.383	No
12.2	4	668.75	821	.75	.250	No
4.3	5	578.0	581	1.07	2.533	No
8.3	5	377.0	385	1.07	.70	No
9.3	5	513.0	521	1.10	.517	No
10.3	5	232.0	240	1.07	.433	No
12.3	5	855.40	893	1.07	.450	No
4.4	6	636.0	636	1.43	2.283	Yes
9.4	6	673.0	678	1.50	1.650	No
10.4	6	313.50	327	1.52	1.0	No
12.4	6	1099.0	1185	1.52	1.050	No
8.5	7	568.14	572	2.35	3.517	No
9.5	7	747.43	750	2.40	2.10	No
10.5	7	346.0	362	2.30	1.267	No
12.5	7	1245.86	1268	2.37	1.850	No

Table IV-2 (cont.)

Number	Periods	LP	IP	LP	IP	Step 8
4.6	8	781.0	786	2.87	35.583	No
8.6	8	734.0	735	3.05	19.60	No
9.6	8	882.50	884	3.12	3.867	No
10.6	8	421.0	422	3.08	1.817	No
12.6	8	1449.0	1534	3.05	4.233	No
8.7	9	843.25	844	3.83	4.550	Yes
9.7	9	1101.25	1102	3.83	16.433	Yes
10.7	9	523.0	525	3.80	7.20	No
12.7	9	1738.89	1844	3.90	21.867	No
4.8	10	921.75	929*	5.43	10^+ min	No
8.8	10	1022.75	1023	5.62	27.067	Yes
9.8	10	1261.50	1263	5.48	287.550	No
10.8	10	603.25	604	5.45	3.817	Yes
12.8	10	1917.50	1965	5.67	41.50	No

Table IV-3

Computational results for the CIDPI algorithm with general demand

Problem Number	Periods (N)	Optimal Values		Run Time(sec)		Step 8 Termination
		LP	IP	LP	IP	
4.1	3	299.33	302	.57	1.067	No
9.1	3	373.0	387	.57	.633	No
4.2	4	472.0	472	1.12	10.80	Yes
8.2	4	271.0	276	1.13	4.650	No
9.2	4	415.67	429	1.13	9.217	No
12.2	4	668.75	728	1.15	2.267	No
4.3	5	578.0	581*	1.73	9^+ min	
9.3	5	513.0	521	1.77	255.350	No
10.3	5	232.0	240	1.73	28.167	No
12.3	5	855.4	868	1.73	38.350	No
9.4	6	673.0	695*	2.55	8^+ min	
10.4	6	313.5	327	2.53	616.980	No
12.4	6	1099.0	1113	2.53	880.0	No

IV.5 Conclusions

The algorithms developed in this thesis take advantage of the structure inherent in the combined capacity expansion and production-inventory problems when no backordering is allowed. The decomposition approach of handling the fixed-charge (capacity expansion) and variable cost (production-inventory) portions of the problem separately allowed the application of special techniques for each part which could then be combined into an efficient algorithm. The regular behavior of the $\bar{y}$ variables in the fixed cost portion of the models allowed the development of useful optimal cost convexity properties in the variable cost section and in the overall problem.

The main problem of integer programming algorithms has not, unfortunately, been avoided in this work. That is, these algorithms display the same unpredictability in running time displayed by all known integer algorithms. Furthermore, computational experience has shown that the algorithm running times are sensitive to demand and cost structures as well as being dependent on the number of periods involved.

The flexibility provided by the restart capability of the algorithms is one of their most

attractive features from a management point of view. The feature allows for efficient use of the linear programming solution initially, by permitting a search among all rounded-off values. Available computer time can then be used efficiently for the main algorithm by running the program for short periods of time, since no information is lost when the enumeration is stopped.

The economic problem of lumpiness in capacity acquisition has been handled rather easily in the extensions of Chapter III. In fact, the more severe the lumpiness is, i.e. the higher the value of k, the larger the intervals in $\bar{y}$-space where the enumeration is performed.

Finally, the observation of an equivalence between the CPI and CIDPI models and the employment smoothing literature is of no small significance. The amount of work done in employment smoothing has been fairly extensive and unification of the results of these two areas would enhance the existing results in both.

IV.6 Extensions

The restrictive assumptions which led to the

results of this thesis provide for some obvious extensions. Perhaps the one with the greatest usefulness in the management context would be a provision for backordering, where the backordering horizon could be restricted to M periods, for $M \varepsilon \{1,2,3,\ldots\}$. Unfortunately, incorporation of backordering in the model will destroy the structure of Standard Transformation 1 and, in addition, if a decomposition approach is taken to solve the problem, the size of the enumeration space would be greatly enlarged. On the surface it seems as if an entirely new approach must be developed for this extension.

Another useful extension would be the development of a model which could handle concave costs of capacity expansion, i.e. decreasing marginal cost in the addition of capacity in any period. The optimal policy results of Manne and Veinott for capacity expansion with concave costs (Manne-Veinott 67) might be useful here.

As was mentioned in Chapter I, the work in capital budgeting is just one step up in aggregation from the content of this thesis. A simple extension in the context of the work already done would be to have a capacity expansion budget in each period. This adds N budget constraints to the models and these

constraints could be handled easily in the fixed charge (capacity) part of the enumeration algorithms.

An extension to many production lines with an overall capacity budget in each period would not be so easy, even if each line had a single product. The difficulty will spring from the problem of optimally allocating the capacity resources among lines in a period as well as optimizing for each line over N periods. A very large-scale integer program will certainly result.

A smaller scale, but equally interesting, problem would be an extension to multiple products on the same production line. As was noted in Chapter III, the work of this thesis limits k to a single value for any model because of the form of Standard Transformation 1. Multiple products are sure to imply different production rates and, hence, different values of k. The challenge appears interesting.

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VITA

Richard H. Barchi, a resident of Yardley, Pennsylvania, was born on 8 November 1942 in Washington, D.C. His undergraduate work was done at the United States Naval Academy where he was awarded the B.S. in 1966. His final year at the Naval Academy was spent in research in Fluid Mechanics on the topic of Boundary Layer Turbulence and Drag Reducing Additives.[1]

After graduation, he served two years as Missile Officer in USS Joseph Strauss (DDG-16) stationed at Pearl Harbor, Hawaii. Matriculating to the Department of Operations Research of The Johns Hopkins University in 1968, the author completed the requirements for the Ph. D. in March of 1972, and returned to duty at sea with the U.S. Navy.

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